Automatic Control- 01
Laplace Transform

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Who is who Automatic Control

- **Instructor**:
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TOPICS

- » Control Systems
- » Laplace Transform
- » System Modeling
- » Time Domain analysis
- » Closed-Loop Control System
- » System Stability
- » Root Locus

Lectures

- Please be courteous in class
- Arrive on time
- Turn off cell phones / sound on laptop
- Sit on the side if you know you have to leave early
- Don’t eat or drink
- Attendance is important
- There are just things that you cannot learn from reading notes

HOMEWORK, DISCUSSIONS, LABS

- Homework assigned weekly
- Very important!
  - Homework due before class on due date
  - » Late homework is not accepted
  - Four laboratory projects
  - Check lab schedule on web page
  - Lab sessions start next week

AGENDA

» Introduction

» Euler's Theorem

» Laplace Transform

» Partial Fraction Expansion

» Using MATLAB to find PFE

» Examples

» Problems

INTRODUCTION

- The Laplace transform method is an operational method that can be used advantageously for solving linear differential equations.
- By use of Laplace transforms, we can convert many common functions, such as sinusoidal functions, damped sinusoidal functions, and exponential functions into algebraic functions of complex variables
- differentiation and integration can be replaced by algebraic operations in the complex plane. Thus, a linear differential equation can be transformed into an algebraic equation in a complex variable **s**

Laplace Merits !!!

- Laplace transform method allows the use of graphical techniques for predicting the system performance without actually solving system differential equations.
- when we solve the differential equation, both the transient component and steady-state component of the solution can be obtained simultaneously

Complex Variable

» A complex number has a **real part** and an imaginary part, both of which are constant.

» If the real part and/or imaginary part are variables, a complex number is called a complex variable.

» In the Laplace transformation we use the notation as a complex variable; that is,

$$\boxed{s = \sigma + j\omega}$$

Where σ is the real part and ω is the imaginary part.

Euler's Theorem

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos\theta + j\sin\theta = e^{j\theta}$$

$$\boxed{\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})}$$

$$\boxed{\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})}$$

Laplace Transform

$$\ell[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

Common Laplace transform pairs

	Time function $f(t)$	Laplace transform $\mathscr{L}[f(t)] = F(s)$
1	unit impulse $\delta(t)$	1
2	unit step 1	$1/s$
3	unit ramp t	$1/s^2$
4	t^n	$\frac{n!}{s^{n+1}}$
5	e^{-at}	$\frac{1}{(s+a)}$
6	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
7	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
8	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
9	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
10	$e^{-at}(\cos \omega t - \frac{a}{\omega} \sin \omega t)$	$\frac{s}{(s+a)^2+\omega^2}$

Laplace Transform

- Constant Multiplication $\longrightarrow$ $\ell\left[af(t)\right]=aF(s)$
- Real shift $\longrightarrow$ $\ell\left[f(t-T)\right]=e^{-Ts}F(s) \quad T\geq 0$
- Convolution Integral $\longrightarrow$ $\int_0^t f_1(\tau)f_2(t-\tau)d\tau=F_1(s)F_2(s)$
- Initial Value Theorem $\longrightarrow$ $f(0)=\lim_{t\to 0}\left[f(t)\right]=\lim_{s\to\infty}\left[sF(s)\right]$
- Final Value Theorem $\longrightarrow$ $f(\infty)=\lim_{t\to\infty}\left[f(t)\right]=\lim_{s\to 0}\left[sF(s)\right]$
- Real differentiation $\longrightarrow$

$$\boxed{\ell\left[\frac{d^n}{dt^n}f(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)}$$

Laplace Transform

$$\ell\left[\int_0^t f(\tau)d\tau\right]=\frac{F(s)}{s}$$

$$\ell\left[t^n f(t)\right]=(-1)^n\frac{d^n}{ds^n}F(s)\qquad for\quad n=1,2,3,\ldots$$

$$\ell\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right]=F_1(s)F_2(s)$$

Common Partial Fraction

$$\frac{k}{s(s+a)} = \frac{A}{s} + \frac{B}{(s+a)}$$

$$\frac{K}{s^2(s+a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+a)}$$

$$\frac{K}{s(as^2+bs+c)} = \frac{K}{s(s+d)(s+e)} = \frac{A}{s} + \frac{B}{(s+d)} + \frac{C}{(s+e)} \rightarrow (b^2 > 4ac)$$

Second-order real roots $(b^2 < 4ac)$

$$\boxed{\frac{K}{s(as^2+bs+c)} = \frac{A}{s} + \frac{Bs+C}{as^2+bs+c}}$$

Completing the square gives

$$\boxed{\frac{A}{s} + \frac{Bs+C}{(s+\alpha)^2+\omega^2}}$$

$f(t)$	$F(s)$
Unit impulse $\delta(t)$	1
Unit step $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!} t^{n-1}e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$

$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s + a)}$
$\frac{1}{b - a}(e^{-at} - e^{-bt})$	$\frac{1}{(s + a)(s + b)}$
$\frac{1}{b - a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s + a)(s + b)}$
$\frac{1}{ab}\left[1 + \frac{1}{a - b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s + a)(s + b)}$

Examples

Example: Find the inverse Laplace transform of

$$\frac{s+3}{(s+1)(s+2)}$$

Solution

$$F(s)=\frac{s+3}{(s+1)(s+2)}=\frac{2}{(s+1)}+\frac{-1}{(s+2)}$$

$$\therefore f(t)=2e^{-t}-e^{-2t}, \quad for \quad t\geq 0$$

Examples

Example: Find the inverse Laplace transform of

$$F(s) = \frac{2s+12}{s^2+2s+5}$$

Solution

$$F(s) = \frac{2s+12}{s^2+2s+5} = \frac{10+2(s+1)}{(s+1)^2+2^2}$$

$$F(s) = 5\frac{2}{(s+1)^2+2^2} + 2\frac{s+1}{(s+1)^2+2^2}$$

$$f(t) = \ell^{-1}\left[F(s)\right]$$

$$f(t) = 5\ell^{-1}\left[\frac{2}{(s+1)^2+2^2}\right] + 2\ell^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right]$$

$$f(t) = 5e^{-t}\sin 2t + 2e^{-t}\cos 2t, \quad for \quad t \geq 0$$

PFE - MATLAB

$$\frac{B(s)}{A(s)} = \frac{num}{den} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

$$num = [b_0 \quad b_1 \quad \ldots \quad b_n]$$

$$den = [a_0 \quad a_1 \quad \ldots \quad a_n]$$

The command

$$[r, p, k] = \text{residue}(\text{num}, \text{den})$$

finds the residues, poles, and direct terms of a partial-fraction expansion of the ratio of two polynomials $B(s)$ and $A(s)$.

The partial-fraction expansion of $B(s)/A(s)$ is given by

$$\frac{B(s)}{A(s)} = \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \ldots + \frac{r(n)}{s - p(n)} + k(s)$$

Example

Example: consider the following transfer function

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

Solution

For this function,

```
>> num=[2 5 3 6];
>> den=[1 6 11 6];
```

The command

```
>> [r,p,k]=residue(num,den)
```

Example

Gives the following result

```
r =

   -6.0000
   -4.0000
    3.0000


p =

   -3.0000
   -2.0000
   -1.0000


k =

     2
```

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$
$$= \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1} + 2$$

The command

$$[num,den] = residue(r,p,k)$$

Where r , p , and k are as given in the previous **MATLAB** output converts the partial-fraction expansion back to the polynomial ratio $B(s)/A(s)$ as follows

```
>> [num,den]=residue(r,p,k)

num =

    2.0000    5.0000    3.0000    6.0000


den =

    1.0000    6.0000   11.0000    6.0000
```

Example: find the partial-fraction expansion of

$$\frac{s+2}{s^3(s-1)^2}$$

Solution

$$\frac{s+2}{s^3(s-1)^2}=\frac{2}{s^3}+\frac{A}{s^2}+\frac{B}{s}+\frac{3}{(s-1)^2}+\frac{C}{(s-1)}$$

$$\therefore s+2=2(s-1)^2+As(s-1)^2+Bs^2(s-1)^2+3s^3+Cs^3(s-1)$$

Coefficient of $s^4=0=B+C \quad \rightarrow \quad I$

Coefficient of $s^3=0=A-2B+3-C \quad \rightarrow \quad II$

Coefficient of $s^2=0=2-2A+B \quad \rightarrow \quad III$

From I, $B = -C$, put it in II,

$$\therefore A - B = -3\ ,$$

and solve it with III,

$$\therefore -2A + B = -2$$

$$\therefore -A = -5 \Rightarrow A = 5, B = 8, C = -8$$

$$\therefore \frac{s+2}{s^3(s-1)^2} = \frac{2}{s^3} + \frac{5}{s^2} + \frac{8}{s} + \frac{3}{(s-1)^2} - \frac{8}{(s-1)}$$

Problems

Example: find the partial-fraction expansion of

$$\frac{s^2+3}{(s+1)(s-2)(s-4)}$$

Example: find the partial-fraction expansion of

$$\frac{3s^2+9s+16}{(s+1)(s^2+4s+13)}$$

Differential equations

Example: find the solution $y(t)$ of the differential equation $y'''(t) - 6y''(t) + 11y'(t) - 6y(t) = 1$

Given $y(0) = y'(0) = y''(0) = 0$

Solution

$$s^3 Y(s) - 6s^2 Y(s) + 11sY(s) - 6Y(s) = 1/s$$

$$\therefore Y(s)\left[s^3 - 6s^2 + 11s - 6\right] = 1/s$$

$$\therefore Y = \frac{1}{s\left(s^3 - 6s^2 + 11s - 6\right)}$$

$$= \frac{1}{s(s-1)(s-2)(s-3)} = \frac{-1/6}{s} + \frac{1/2}{s-1} - \frac{1/2}{s-2} + \frac{1/6}{s-3}$$

$$\therefore y(t) = -\frac{1}{6} + \frac{1}{2}e^{t} - \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t}$$

Example: find the solution $y(t)$ of the differential equation

$$y''(t) + 2y'(t) + y(t) = t$$
$$y(0) = -3 \qquad y(1) = -1$$

Example: find the solution $x(t)$ of the differential equation

$$x''(t) + 4x(t) = 10\sin 3t$$
$$x(0) = x'(0) = 0$$

Example: find the solution $x(t)$ of the differential equation

$$y'''(t) + y(t) = 1$$
$$y(0) = y'(0) = y''(0) = 0$$