Chapter 03

Laplace Transform

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■ Laplace transforms

- $F(s) = \mathcal{L}\, f(t) = \int_0^{\infty} e^{-st} f(t)\, dt$
- This technique transforms the problem from the time (t) domain to the Laplace (s) domain. The advantage in doing this is that complex time domain differential equations become relatively simple s domain algebraic equations. When a suitable solution is arrived at, it is inverse transformed back to the time domain.
- where s is a complex variable $\sigma \pm j\omega$ and ℓ is called the Laplace operator

Example

Unit-step input $f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases}$

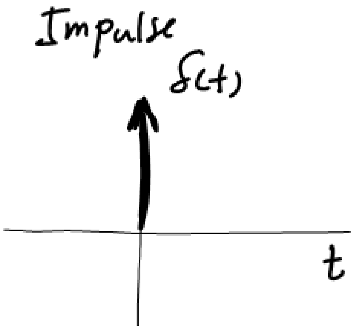
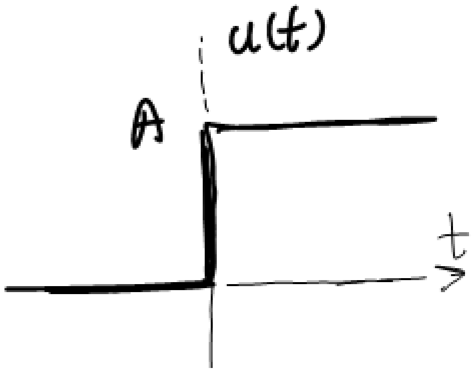
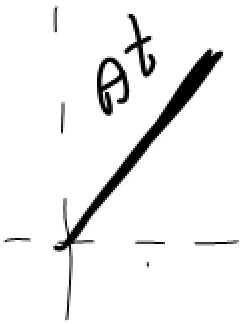
$$F(s) = \int_0^{\infty} e^{-st}\, 1\, dt = \left. \frac{e^{-st}}{-s} \right|_{\infty}^{0} = 1/s$$

Example

Unit-step input $f(t) = \begin{cases} a & t \geq 0 \\ 0 & otherwise \end{cases}$

$$F(s) = \int_0^{\infty} e^{-st}\, a\, dt = a \left. \frac{e^{-st}}{-s} \right|_{\infty}^{0} = a/s$$

- **Test functions**

Impulse $\delta(t)$	
Unit-step $u(t)$	
Ramp $f(t) = At$	

Common Laplace transform pairs

	Time function $f(t)$	Laplace transform $\mathscr{L}[f(t)] = F(s)$
1	unit impulse $\delta(t)$	1
2	unit step 1	$1/s$
3	unit ramp t	$1/s^2$
4	t^n	$\dfrac{n!}{s^{n+1}}$
5	e^{-at}	$\dfrac{1}{(s+a)}$
6	$1 - e^{-at}$	$\dfrac{a}{s(s+a)}$
7	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
8	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
9	$e^{-at} \sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
10	$e^{-at}(\cos \omega t - \dfrac{a}{\omega} \sin \omega t)$	$\dfrac{s}{(s+a)^2+\omega^2}$

Important properties of Laplace transform

- **Constant multiplication**

$$\mathcal{L}\,[af(t)] = a\,F(s)$$

- **Real-shift theorem**

$$\mathcal{L}\,f(t-T) = e^{-sT}F(s) \quad T \geq 0$$

- **Convolution Integral**

$$\int_0^t f_1(\tau) f_2(t-\tau) \quad d\tau = F_1(s)F_2(s)$$

- **Initial value theorem**

$$f(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$$

- **Final value theorem**

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- **Real differentiation theorem**

$$\mathcal{L}\,\frac{d^n}{dt^n} f(t) = s^n F(s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0) - s^{n-3} \ddot{f}(0) \dots f(0)^{n-1}$$

- **Integration**

$$\mathcal{L}\int_0^t f(\tau)\,d\tau = \frac{F(s)}{s}$$

- **Multiplication by t^n**

$$\mathcal{L}[t^n f(t)] = -1^n \frac{d^n}{ds^n} F(s) \quad for\; n = 1,2,3,\dots$$

Common partial fraction expansion

- Factored roots

$$\frac{k}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

- Repeated roots

$$\frac{k}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

- Second-order real roots $b^2 > 4ac$

$$\frac{K}{s(as^2+bs+c)} = \frac{K}{s(s+d)(s+e)} = \frac{A}{s} + \frac{B}{s+d} + \frac{C}{s+e}$$

- Second-order real roots $b^2 < 4ac$

$$\frac{K}{s(as^2+bs+c)} = \frac{A}{s} + \frac{Bs+C}{as^2+bs+c}$$

Completing the squares, gives

$$\frac{A}{s}+\frac{Bs+C}{(s+\alpha)^2+\omega^2}$$

$f(t)$	$F(s)$
Unit impulse $\delta(t)$	1
Unit step $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!} \quad (n=1,2,3,\dots)$	$\frac{1}{s^n}$
$t^n \quad (n=1,2,3,\dots)$	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at} \quad (n=1,2,3,\dots)$	$\frac{1}{(s+a)^n}$
$t^n e^{-at} \quad (n=1,2,3,\dots)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
$1 - \cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

Find Laplace transform of

$$f(t) = e^{-at}\sin\omega t$$

$\because \quad \sin\omega t = \dfrac{\omega}{s^2+\omega^2}$

$\therefore \quad F(s) = \dfrac{\omega}{(s+a)^2+\omega^2}$

Given that:

$$Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Find $y(t)$

By partial fraction

$$Y(s) = \frac{8/3}{s} - \frac{3/2}{s+1} - \frac{1/6}{s+3}$$

$$y(t) = \frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Find the inverse Laplace of

$$\frac{s+3}{(s+1)(s+2)}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} + \frac{-1}{(s+2)}$$

$\ell^{-1} = 2e^{-t} - e^{-2t}$

Find the inverse Laplace of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

$$F(s) = \frac{2(s+1) + 10}{(s+1)^2 + 4}$$

$$F(s) = 2\frac{(s+1)}{(s+1)^2 + 4} + 5\frac{2}{(s+1)^2 + 4}$$

$$f(t) = 2e^{-t}\cos 2t + 5e^{-t}\sin 2t$$

Find the solution to the differential equation

$\ddot{y}(t) + y(t) = 0$ where $y(0) = \alpha$, $\dot{y}(0) = \beta$

$$s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = 0$$

$$Y(s)[s^2 + 1] = \alpha s + \beta$$

$$Y(s) = \frac{\alpha s + \beta}{s^2 + 1}$$

$$y(t) = \alpha\cos t + \beta\sin t$$

Find the solution to the differential equation

$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = u(t)$ where $y(0) = 0$, $\dot{y}(0) = 0$

$u(t) = 2e^{-2t}$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 5[sY(s) - y(0)] + 4Y(s) = \frac{2}{s+2}$$

$$Y(s)[s^2 + 5s + 4] = \frac{2}{s+2}$$

$$Y(s) = \frac{2}{(s+2)(s^2+5s+4)} = \frac{2}{(s+2)(s+1)(s+4)}$$

$$Y(s) = \frac{2/3}{s+1} - \frac{1}{s+2} + \frac{1/3}{s+4}$$

$$y(t) = \frac{2}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t}$$

Find inverse Laplace of

$$Y(s) = \frac{s+2}{s^3(s-1)^2}$$

$$\frac{s+2}{s^3(s-1)^2} = \frac{2}{s^3} + \frac{A}{s^2} + \frac{B}{s} + \frac{3}{(s-1)^2} + \frac{C}{s-1}$$

$$\therefore \quad \frac{s+2}{s^3(s-1)^2} = \frac{2}{s^3} + \frac{5}{s^2} + \frac{8}{s} + \frac{3}{(s-1)^2} - \frac{8}{s-1}$$

$$\therefore \quad y(t) = t^2 + 5t + 8 + 3te^t - 8e^t$$

Find inverse Laplace of

$$Y(s) = \frac{3s^2 + 9s + 16}{(s+1)(s^2 + 4s + 13)}$$

$$Y(s) = \frac{1}{s+1} + \frac{As + B}{s^2 + 4s + 13}$$

$$Y(s) = \frac{1}{s+1} + \frac{2s + 3}{s^2 + 4s + 13}$$

$$Y(s) = \frac{1}{s+1} + \frac{2(s+2) - 1}{(s+2)^2 + 9}$$

$$Y(s) = \frac{1}{s+1} + 2\frac{(s+2)}{(s+2)^2 + 9} + \frac{-1}{(s+2)^2 + 9}$$

$$y(t) = e^{-t} + 2e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$$

Given that:

$$\dddot{y}(t) - 6\ddot{y}(t) + 11\dot{y}(t) - 6y(t) = 1$$

$$y(0) = \dot{y}(0) = \ddot{y}(0) = 0$$

Ans.

$$y(t) = -\frac{1}{6} + \frac{1}{2}e^t - \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t}$$

Given that: $\ddot{y}(t) + 2\dot{y}(t) + y(t) = t$ $y(0) = -3 \qquad y(1) = -1$
Ans. Assume $\dot{y}(0) = A$ $y(t) = (t-1)e^{-t} + t - 2$

Given that: $\ddot{x}(t) + 4x(t) = 10\sin 3t$ $x(0) = \dot{x}(0) = 0$

$$s^2X(s) - sx(0) - \dot{x}(0) + 4X(s) = \frac{30}{s^2+9}$$

$$X(s)(s^2+4) = \frac{30}{s^2+9}$$

$$X(s) = \frac{30}{(s^2+4)(s^2+9)}$$

$$X(s) = \frac{6}{(s^2+4)} - \frac{6}{(s^2+9)}$$

$$x(t) = 3\sin 2t - 2\sin 3t$$

Given that:

$\ddot{y}(t) + y(t) = 1$

$y(0) = \dot{y}(0) = \ddot{y}(0) = 0$

Find $y(t)$

- Final value theorem

Suppose we have a transform $Y(s)$ of a signal $y(t)$ and wish to know the final value $y(t)$ from $Y(s)$: there are three possibilities

1. $Y(s)$ has any **poles** in the **RHS** (Right hand side) — that is, if real part of any $P_i > 0$ then, $y(t)$ will grow and the limit will be unbounded
2. $Y(s)$ has a pair of poles on the imaginary axis of the s-plane (i.e. $P_i = \pm j\omega$) then, $y(t)$ will have a sinusoid that persist forever and the final value will not be defined
3. All poles of $Y(s)$ are in LHS (Left Hand Side), except for one at $s = 0$, then all terms of $y(t)$ will decay to zero except the term corresponding to the pole at $s = 0$, and that term corresponds to a constant in time

Find the final value of the system corresponds to

$$Y(s) = \frac{3(s+2)}{s(s^2+2s+10)}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$\lim_{t\to\infty} y(t) = 6/10$$

Thus, after the transients have decayed to zero, $y(t)$ will settle to a constant value of 0.6

Note:

Care must be taken to use final value theorem in an unstable system

Find the final value of the system corresponds to

$$Y(s) = \frac{3}{s(s-2)}$$

If we blindly apply the equation of final value theorem, we obtain

$$y(\infty) = -\frac{3}{2}$$

However:

$y(t) = -\frac{3}{2} + \frac{3}{2}e^{2t}$ — unbounded

■ MATLAB session

Residue

- Partial fraction expansion
- **`[r,p,k] = residue(b,a)`** finds the **residues**, **poles**, and **direct term** of a Partial Fraction Expansion of the ratio of two polynomials, where the expansion is of the form

$$\frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{r_n}{s-p_n} + \ldots + \frac{r_2}{s-p_2} + \frac{r_1}{s-p_1} + k(s).$$

- The inputs to **residue** are vectors of coefficients of the polynomials **`b = [bm ... b1 b0]`** and **`a = [an ... a1 a0]`**. The outputs are the residues `r = [rn ... r2 r1]`, the poles **`p = [pn ... p2 p1]`**, and the polynomial `k`. For most textbook problems, `k` is 0 or a constant.
- `[b,a] = residue(r,p,k)` converts the partial fraction expansion back to the ratio of two polynomials and returns the coefficients in b and a.

Find the partial fraction expansion of the following ratio of polynomials $F(s)$ using residue

$$F(s) = \frac{b(s)}{a(s)} = \frac{-4s+8}{s^2+6s+8}$$

```
b = [-4 8]
a = [1 6 8]
[r,p,k] = residue(b,a)
```

```
r =

   -12
     8


p =

    -4
    -2


k =

     []
```

This represents the partial fraction expansion

$$\frac{-4s+8}{s^2+6s+8} = \frac{-12}{s+4} + \frac{8}{s+2}$$

Convert the partial fraction expansion back to polynomial coefficients using `residue`.

```
[b,a] = residue(r,p,k)
```

```
b =

    -4     8

a =
```

```
     1          6          8
```

This result represents the original fraction $F(s)$.

Find partial fraction expansion of

$$Y(s) = \frac{s+2}{s^3(s-1)^2}$$

Hint: use **conv** → Convolution and polynomial multiplication

```
b = [1 2];
a1 = [1 0 0 0];
a2 = [1 -1];
a3 = [1 -1];
a = conv(a1,a2);
a = conv(a,a3);
[r,p,k] = residue(b,a)
```

r =

  -8

   3

   8

   5

   2

p =

   1

   1

0

0

0

k =

[]

$$\frac{s+2}{s^3(s-1)^2} = \frac{-8}{s-1} + \frac{3}{(s-1)^2} + \frac{8}{s} + \frac{5}{s^2} + \frac{2}{s^3}$$

Find the partial fraction expansion of

$$Y(s) = \frac{s^2+3}{(s+1)(s-2)(s-4)}$$

Hint: use **conv** → Convolution and polynomial multiplication

```
b = [1 0 3];
a1 = [1 1];
a2 = [1 -2];
a3 = [1 -4];
a = conv(a1,a2);
a = conv(a,a3);
[r,p,k] = residue(b,a)
```

r =

1.9000

-1.1667

0.2667

p =

4.0000

2.0000

-1.0000

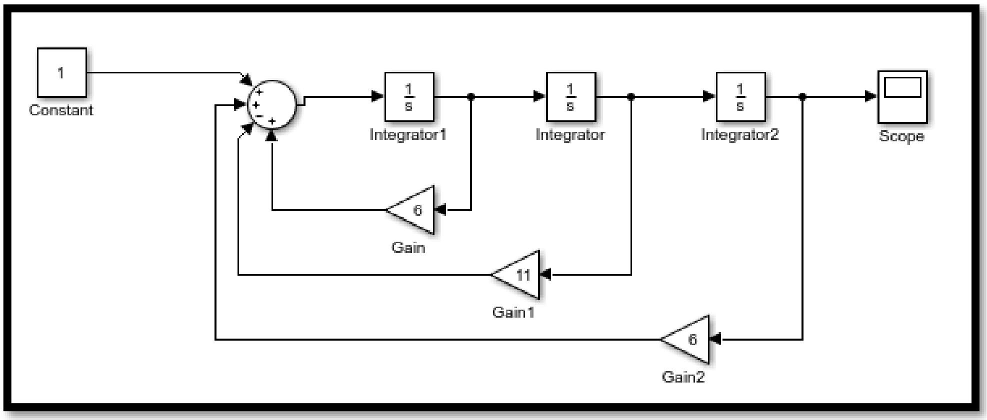
k =

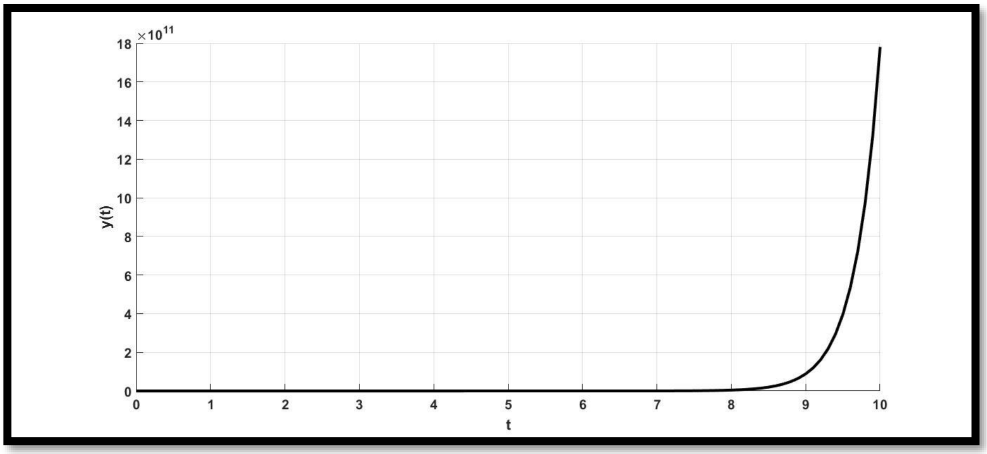
[]

$$\frac{s^2+3}{(s+1)(s-2)(s-4)} = \frac{1.9}{s-4} - \frac{1.1667}{s-2} + \frac{0.226}{s+1}$$

Given that: $\dddot{y}(t) - 6\ddot{y}(t) + 11\dot{y}(t) - 6y(t) = 1$ $y(0) = \dot{y}(0) = \ddot{y}(0) = 0$ Find $y(t)$ by MATLAB and SIMULNIK
Ans. $y(t) = -\frac{1}{6} + \frac{1}{2}e^{t} - \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t}$

```
t = 0:0.1:10;
y = -1/6+0.5*exp(t)-0.5*exp(2*t)+(1/6)*exp(3*t);
plot(t,y)
```

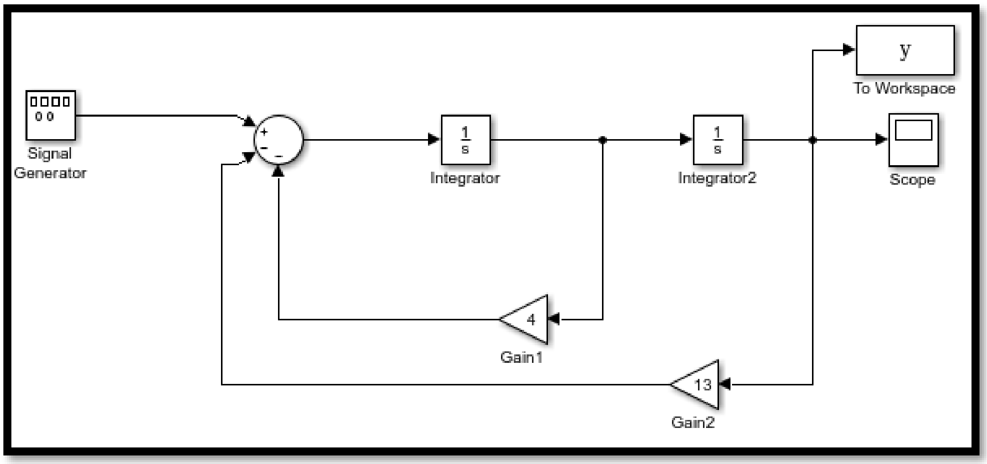


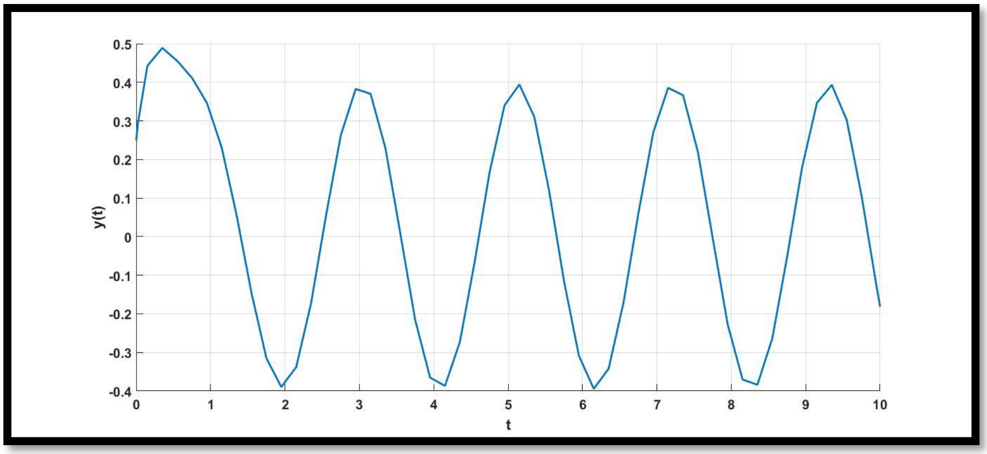


Solve the equation

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 5\cos 3t$$

Given that $y(0) = \frac{1}{4}$, $\dot{y}(0) = 2$





Problems:

1. Solve the equation: $\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 5\cos 3t$

   Given that $y(0) = \frac{1}{4}$, $\dot{y}(0) = 2$

   This is an example of ***forced oscillations with damping***—

   Sketch the Plot of $y(t)$

2. Solve the equation: $\ddot{x}(t) + 2\dot{x}(t) + x(t) = 3te^{-t}$

   Given that $x(0) = 4$, $\dot{x}(0) = 2$