Chapter 04

Modeling

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■ Mathematical model

- If the dynamic behavior of a physical system can be represented by an equation, or a **set of equations**, this is referred to as the mathematical model of the system.
  - Mathematical description of the process to be controlled
  - A set of differential equations that describe the dynamic behavior of the process
- Such models can be constructed from **knowledge of the physical characteristics of the system**, i.e. **mass** for a mechanical system or **resistance** for an electrical system.
- Alternatively, a mathematical model may be determined by **experimentation**, by measuring **how the system output responds to known inputs**.
- In general, consider a system whose output is $\boldsymbol{y(t)}$, whose input is $\boldsymbol{x(t)}$and contains constant coefficients of values a, b, c. If the dynamics of the system produce a **first-order** differential equation, it would be represented as

$$\boldsymbol{a\frac{dy(t)}{dt} + by(t) = cx(t)}$$

- If the system dynamics produced a **second-order** differential equation, it would be represented by

$$\boldsymbol{a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = ex(t)}$$

- If the dynamics produce a **third-order** differential equation, its representation would be

$$\boldsymbol{a\frac{d^3y(t)}{dt^3} + b\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ey(t) = fx(t)}$$

- Equations above are linear differential equations with constant coefficients. Note that the **order of the differential equation** is the **order of the highest derivative**.
- Systems described by such equations are called **linear systems** of the same order as the differential equation. For example, **first-order linear system**, **second-order** linear system and **third-order** linear system.

A control system may be composed of various components including **mechanical**, **thermal**, **fluid**, **pneumatic**, and **electrical**; **sensors** and **actuators**; and **computers**

Using the basic **modeling principles** such as **Newton's second law** of motion or **Kirchhoff's law**, the models of these dynamic systems are represented by **differential equations**. It is not difficult to understand that the analytical and computer simulation of any system is only as good as the model used to describe it. It should also be emphasized that the modern control engineer should place special emphasis on the mathematical modeling of systems so that analysis and design problems can be conveniently solved by computers.

■ Mechanical systems modeling

Mechanical systems may be modeled as **systems of lumped masses** (**rigid bodies**) or as **distributed mass** (continuous) systems. The latter are modeled by **partial differential equations**, whereas the former is represented by **ordinary differential equations**.

Of course, in reality all systems are continuous, but, in most cases, it is easier and therefore preferred to approximate them with lumped mass models and ordinary differential equations

The motion of mechanical elements can be described in various dimensions as **translational**, **rotational**, or a **combination of both**.

The equations governing the motion of mechanical systems are often directly or indirectly formulated from Newton's law of motion.

The equations of a linear mechanical system are written by first constructing a model of the system containing interconnected linear elements and then by applying **Newton's law of motion** to the **free-body diagram** (FBD).

- **Translational motion**

The motion of **translation** is defined as a **motion that takes place along a straight or curved path**. The variables that are used to describe translational motion are **acceleration**, **velocity**, and **displacement**.

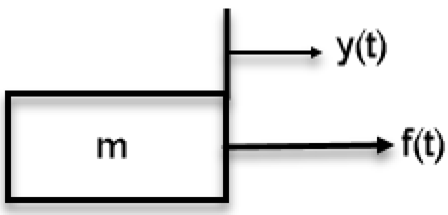
Newton's law of motion states that the algebraic sum of external forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction.

The law can be expressed as

$$\sum_{\boldsymbol{External}} \boldsymbol{Forces} = \boldsymbol{m\,a}$$

where $\boldsymbol{m}$ denotes the mass, and $\boldsymbol{a}$ is the acceleration in the direction considered.

Fig. below illustrates the situation where a force is acting on a body with mass $\boldsymbol{m}$ The force equation is written as



$$f(t) = m\,a(t) = m\frac{d^2y}{dt^2} = m\frac{dv}{dt}$$

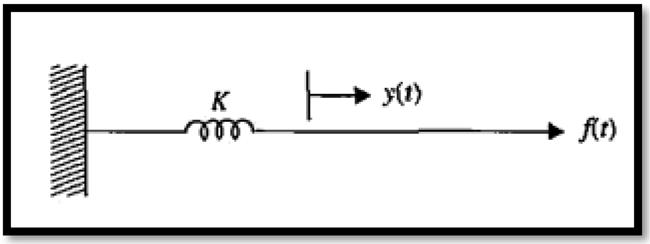
where $\boldsymbol{a}(\boldsymbol{t})$ is the acceleration, $\boldsymbol{v}(\boldsymbol{t})$ denotes linear velocity, and $y(t)$ is the displacement of mass $\boldsymbol{m}$, respectively. For linear translational motion, in addition to the mass, the following system elements are also involved.

Linear spring. In practice, a linear spring may be a model of an actual spring or a compliance of a cable or a belt. In general, a spring is considered to be an element that stores potential energy.

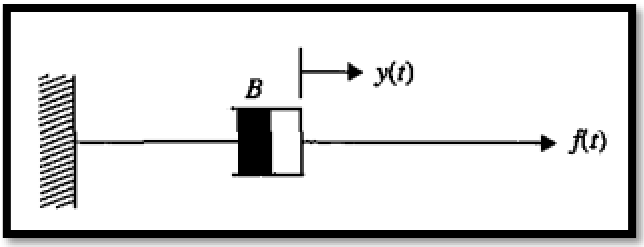
$$f(t) = k\, y(t)$$

where $\boldsymbol{k}$ is the spring constant, or simply stiffness. Equation above implies that the **force acting on the spring is directly proportional to the displacement (deformation) of the spring**.

The model representing a linear spring element is shown in Fig. below.



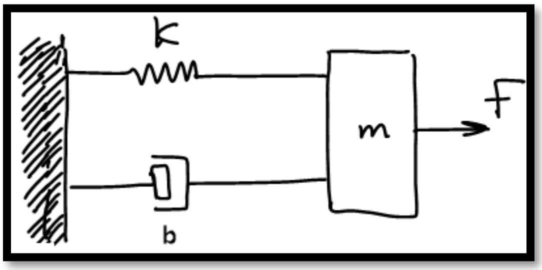
Friction: Whenever there is motion or tendency of motion between two physical elements, frictional forces exist, i.e. that is a **linear relationship between the applied force and velocity**

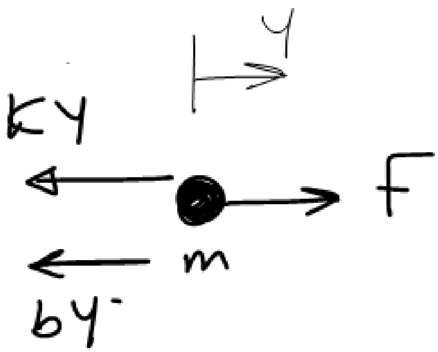


$$f(t) = \beta \frac{dy}{dt}$$

where β is the viscous frictional coefficient

Consider the mass-spring-friction system shown in Fig. below. The linear motion concerned is in the horizontal direction.





$$f(t) - b\dot{y}(t) - ky(t) = m\ddot{y}(t)$$

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = f(t)$$

Taking Laplace, it gives, and assume zero initial condition

$Y(s)[ms^2 + bs + k] = F(s)$

$\boldsymbol{\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + bs + k}}$	Transfer function
▪ $\boldsymbol{m}$, $\boldsymbol{b}$, $\boldsymbol{k}$ are physical parameters ▪ Changing these parameters, also change the **system response**	

Assume $f(t) = u(t) = 1\ N$

$$\therefore \quad F(s) = \frac{1}{s}$$

And $m = 1$, $b = 0.1$, $k = 0.1$

$$\boldsymbol{\frac{Y(s)}{F(s)} = \frac{1}{s^2 + 0.1s + 0.1}}$$

1. **Analytically**

$$Y(s) = \frac{1}{s(s^2 + 0.1s + 0.1)} = \frac{10}{s} + \frac{As + B}{s^2 + 0.1s + 0.1}$$

$$\therefore \quad Y(s) = \frac{10}{s} - \frac{10s + 1}{s^2 + 0.1s + 0.1}$$

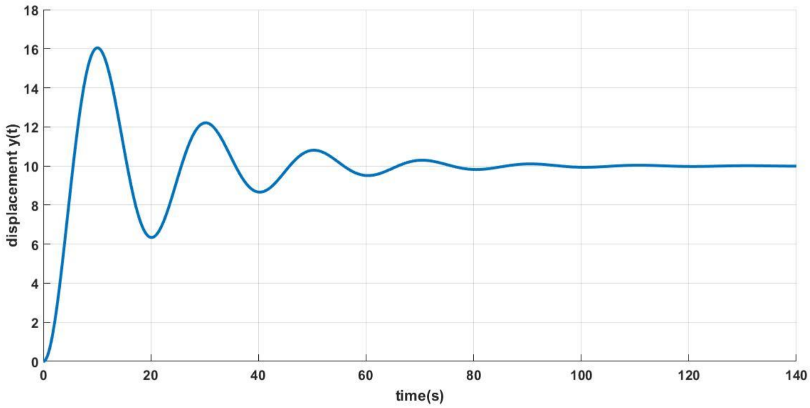
$$Y(s) = \frac{10}{s} - \frac{10(s + 0.05) + 0.5}{(s + 0.05)^2 + 0.0975}$$

$$y(t) = 10 - 10e^{-0.05t}\cos\sqrt{0.0975}t - \frac{0.5}{\sqrt{0.0975}}e^{-0.05t}\sin\sqrt{0.0975}t$$

2. By MATLAB control system toolbox

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{F(s)}} = \frac{\mathbf{1}}{\boldsymbol{s^2 + 0.1s + 0.1}}$$

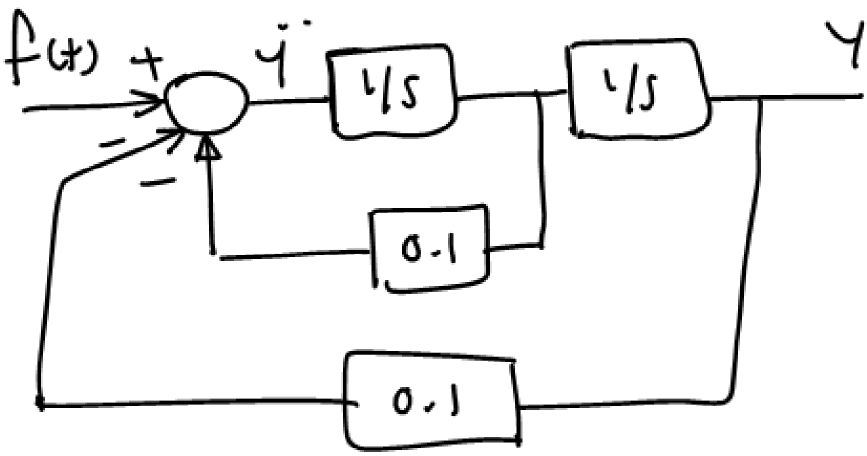
```
b = 0.1;
k = 0.1;
num = [1];
den = [m b k];
t = 0:0.1:140;
sys = tf(num,den);
y = step(sys,t);
plot(t,y);
```

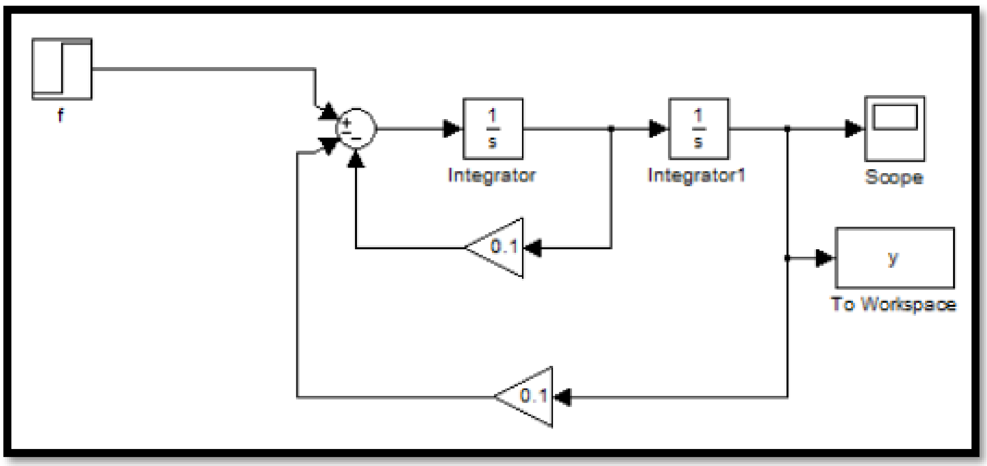


3. By Simulink

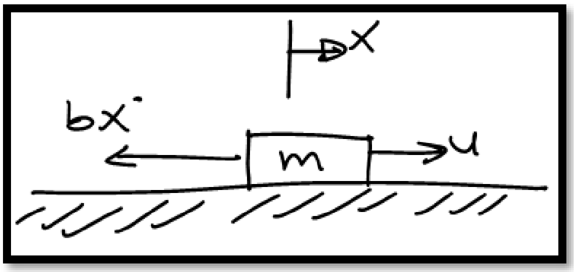
$$\frac{Y(s)}{F(s)} = \frac{1}{s^2 + 0.1s + 0.1}$$

$$\ddot{y} + 0.1\dot{y} + 0.1y = f(t)$$





Write the equation of motion for the speed and forward motion of the car, assuming that the engine impacts a force u as shown, find the transfer function between the input u and the output v



$$u - b\dot{x} = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} = u$$

$$m\dot{v} + bv = u$$

Take Laplace transform

$$(ms + b)V = U$$

$$\frac{V}{U} = \frac{1}{ms + b} = \frac{1/m}{s + b/m}$$

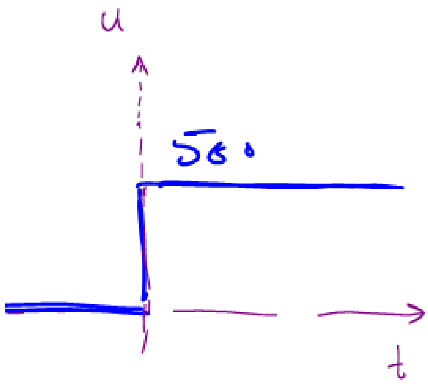
Example:

Use MATLAB to find the response of the velocity of the car for the case in which the input jumps from being $u = 0$ at time $t = 0$ to a constant $u = 500\ N$ thereafter. Assume that the car mass is 1000 kg and viscous drag coefficient $b = 50$

$$\because \quad \frac{V}{U} = \frac{1/m}{s + b/m} = \frac{1/1000}{s + 50/1000}$$

$$\frac{V}{U} = \frac{0.001}{s + 0.05}$$

$$u = \begin{cases} 500 & t \geq 0 \\ 0 & otherwise \end{cases}$$



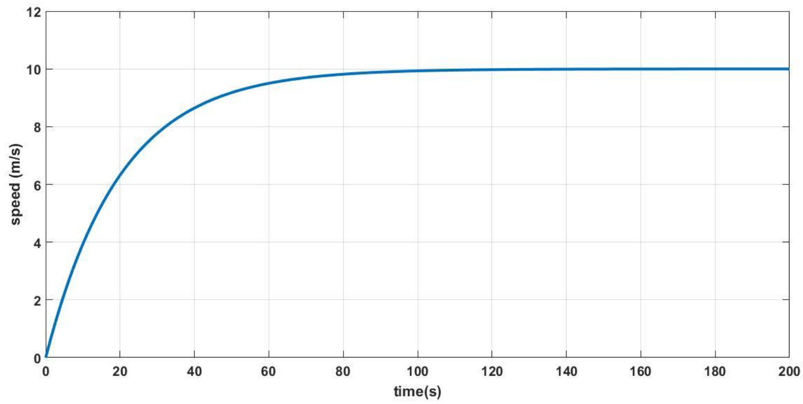
$$U(s) = 500/s$$

$$\therefore \quad V(s) = \frac{0.5}{s(s + 0.05)}$$

Partial fraction

$$V(s) = \frac{10}{s} - \frac{10}{s + 0.05}$$

$$v(t) = 10 - 10e^{-0.05t}$$

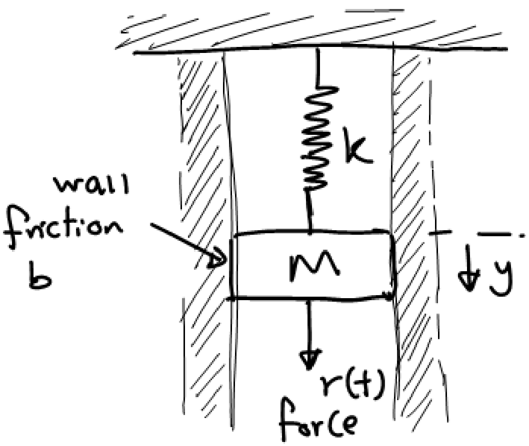


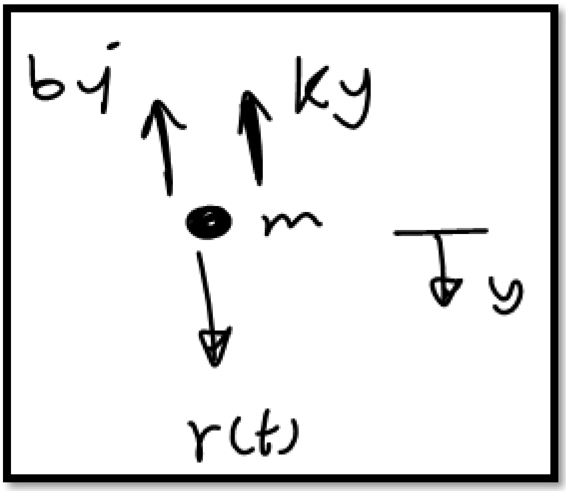
You can also plot using MATLAB toolbox

$$\frac{V}{U} = \frac{0.001}{s + 0.05}$$

```
num = [0.001];
den = [1 0.05];
sys = tf(num,den);
step(500*sys)
```

Example: Spring-mass-damper (automobile shock absorber)





$$m\ddot{y} = r - b\dot{y} - ky$$

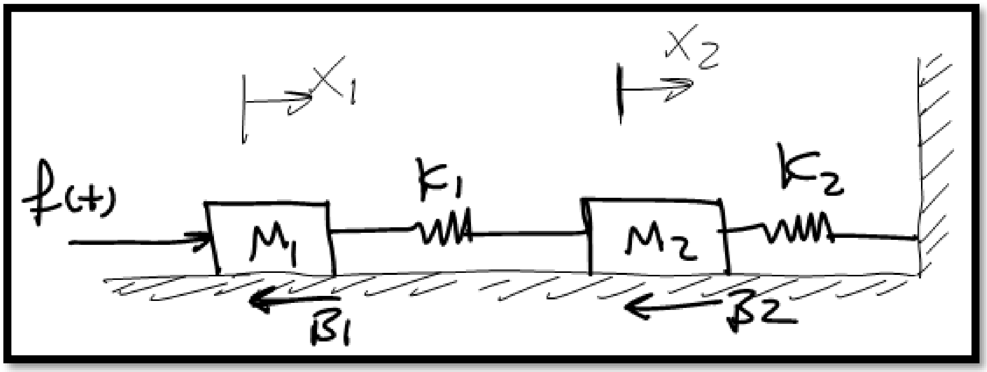
$$m\ddot{y} + b\dot{y} + ky = r$$

$$(ms^2 + bs + k)Y = r$$

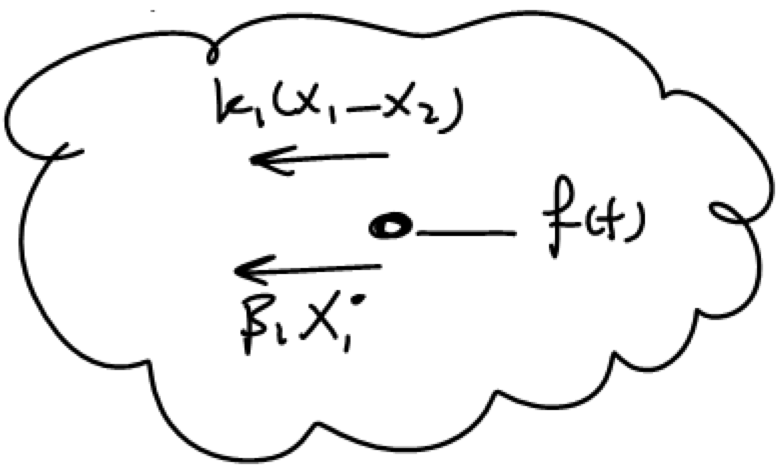
$$\frac{Y}{r} = \frac{1}{ms^2 + bs + k}$$

Changing system parameters (m, b, k) change the value and shape of the output signal

Example:



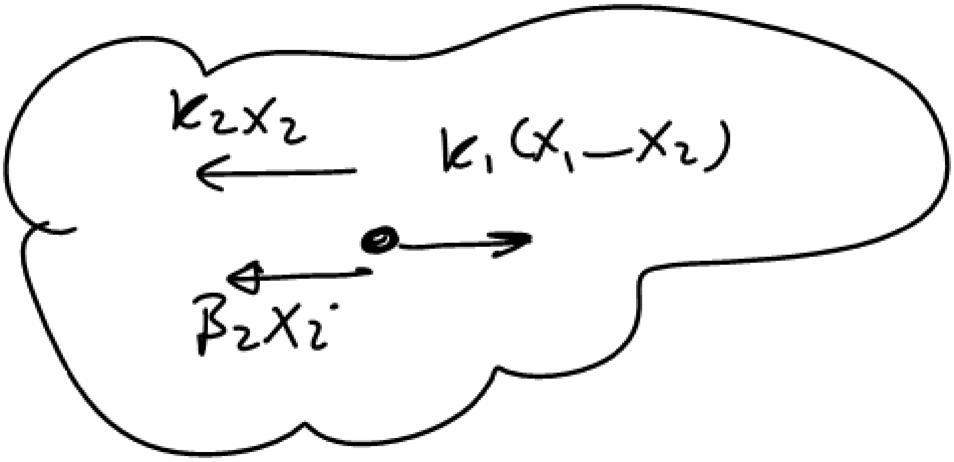
Free body diagram of mass M_1



$$M_1\ddot{x_1} = f - \beta_1\dot{x_1} - k_1(x_1 - x_2)$$

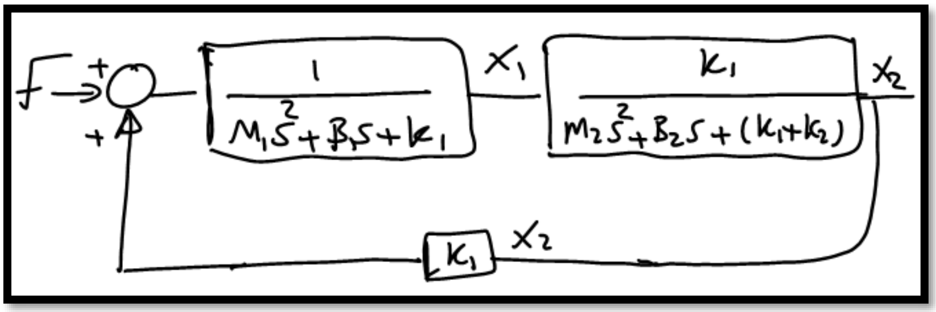
$$\therefore \quad F = (M_1 s^2 + \beta_1 s + k_1)X_1 - k_1 X_2$$

Free body diagram of mass M_2



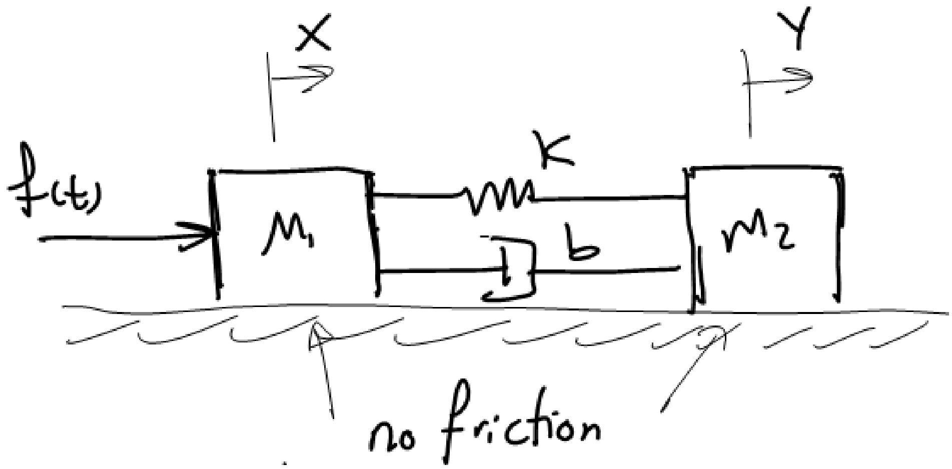
$M_2\ddot{x_2} = k_1(x_1 - x_2) - \beta_2\dot{x_2} - k_2x_2$

$\therefore \quad k_1X_1 = (M_2s^2 + \beta_2 s + k_1 + k_2)X_2$



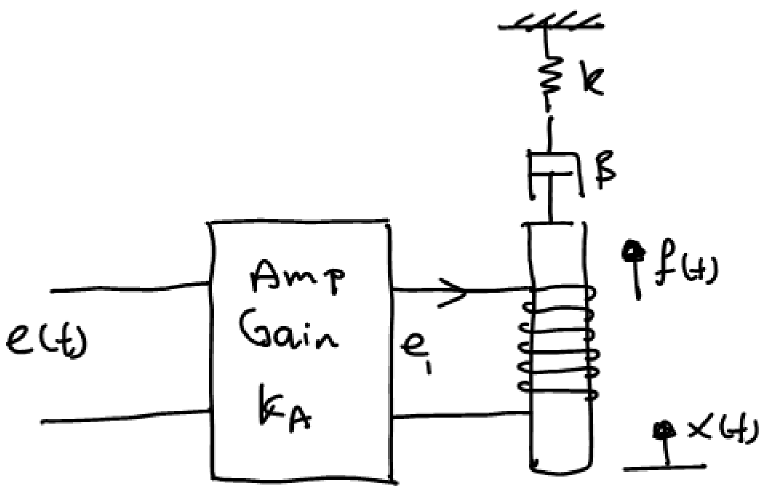
Problem:

Sketch the block diagram for the mechanical system shown below



Example:

Find the transfer function $\frac{X(s)}{E(s)}$

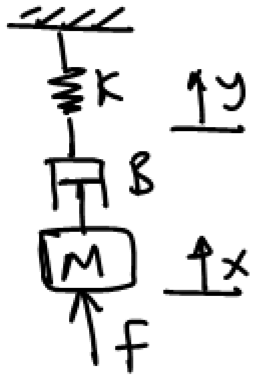


$E_1 = k_A E$

$E_1 = I(R + sL)$

$F = (ms^2 + \beta s)X - \beta sY$

$\beta sY = (\beta s + k)Y$



Please continue

Example:

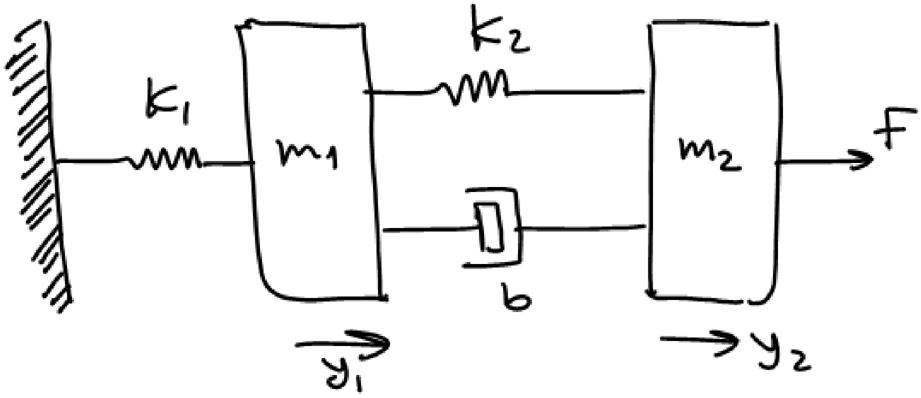
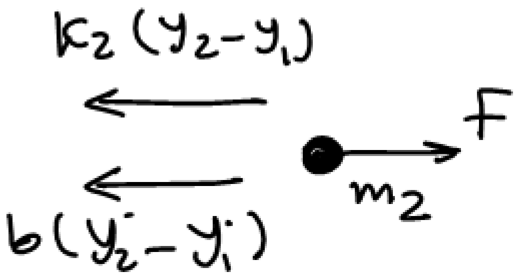
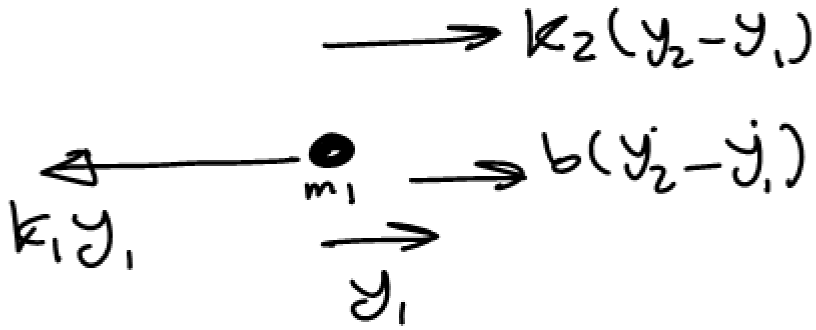


Figure above shows a mechanical system with two masses and two springs. Obtain the mathematical model of the system



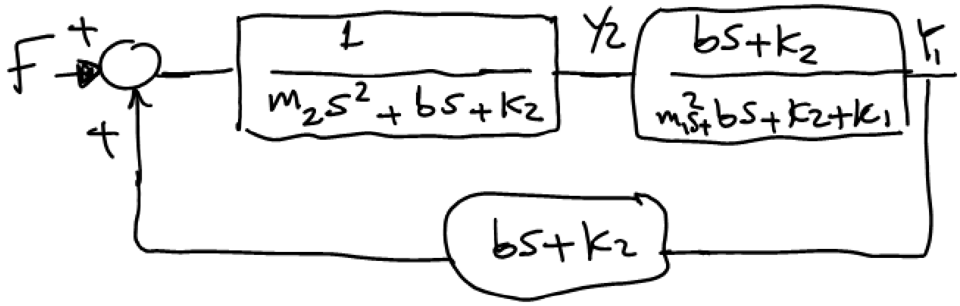
$$f - b(\dot{y}_2 - \dot{y}_1) - k_2(y_2 - y_1) = m_2\ddot{y}_2$$

$$F = (m_2s^2 + bs + k_2)Y_2 - (bs + k_2)Y_1$$

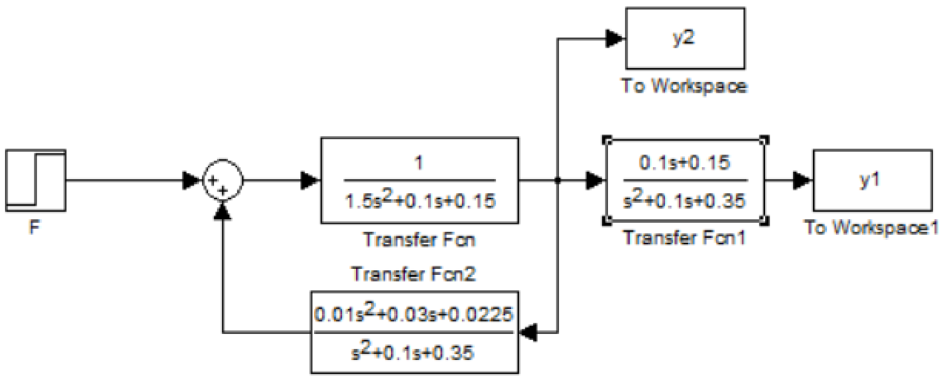


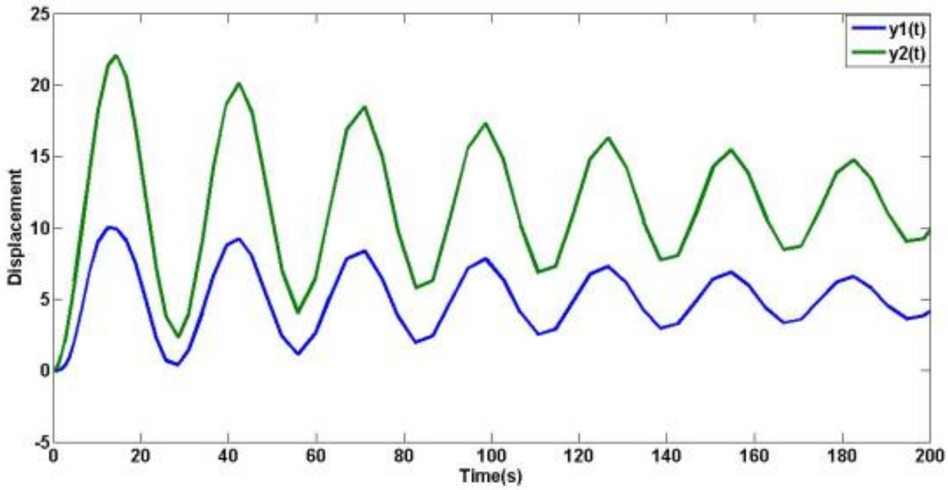
$$b(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - y_1) - k_1 y_1 = m_1 \ddot{y}_1$$

$$(bs + k_2)Y_2 = (m_1 s^2 + bs + k_2 + k_1)Y_1$$



Assume $f = 1$, $m_1 = 1$, $m_2 = 1.5$, $b = 0.1$, $k_1 = 0.2$, $k_2 = 0.15$





■ Modeling in electrical circuit

Electric circuits are frequently used in control systems largely because of the ease of manipulation and processing of electric signals

Although controllers are increasingly implemented with digital logic, many functions are still performed with analog circuit

1. Analog circuits are fasters than digital, and for very simple controllers, an analog circuit would be less expensive than a digital implementation
2. Furthermore, the power amplifier for electromechanical control for digital control must be analog circuit

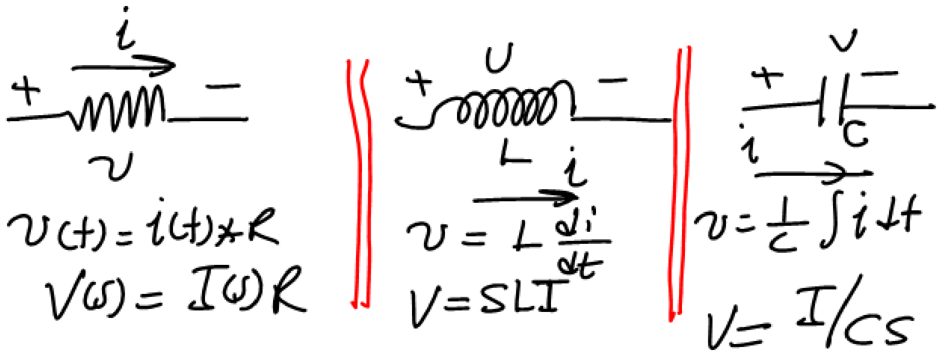
- **Kirchhoff's laws**

Kirchhoff current law: the algebraic sum of currents leaving a junction or node equals the algebraic sum of currents entering that node

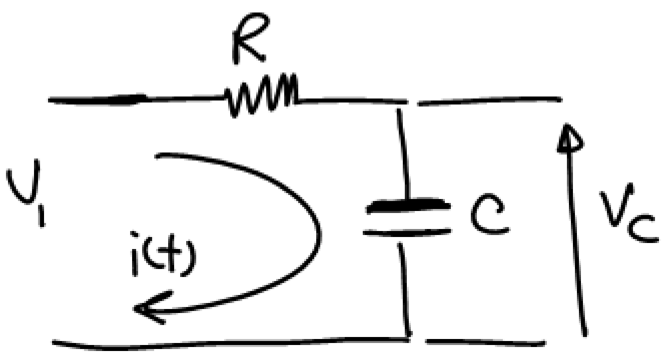
$$\boldsymbol{I_{in} = I_{out}}$$

Kirchhoff voltage law: the algebraic sum of all voltages taken around a closed path in a circuit is zero

$$\sum \boldsymbol{voltage\ sources} = \sum \boldsymbol{voltage\ drops}$$



Example:



$$v_1(t) = i(t)\,R + \frac{1}{C}\int i(t)\,dt$$

Taking Laplace

$$V_1(s) = I(s)\,R + \frac{I(s)}{Cs}$$

$$V_1(s) = I(s)\left(R + \frac{1}{Cs}\right)$$

$$v_c(t) = \frac{1}{C}\int i(t)\,dt$$

$$V_c(s) = \frac{I(s)}{Cs}$$

$$\therefore \quad \frac{V_c}{V_1} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1 + RCs}$$

Mathematically ➔ analytically

Assume $v_1(t) = u(t) = 1\ volt$

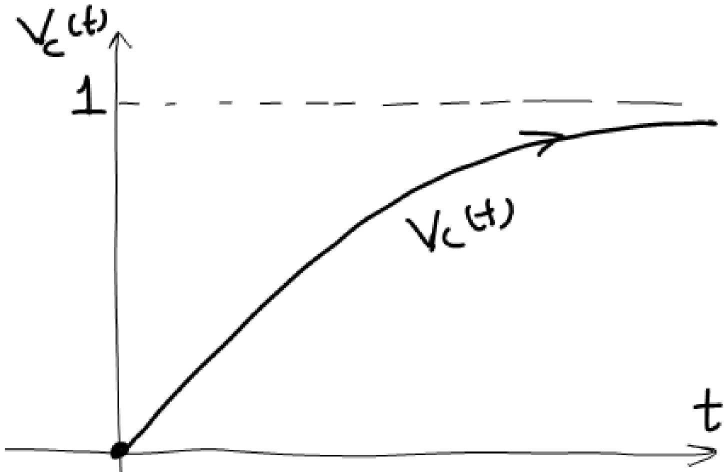
$$V_1 = 1/s$$

$$V_c = \frac{1}{s} - \frac{1}{s + 1/RC}$$

$$v_c(t) = 1 - e^{-t/RC}$$

Assume $R = 1$, $C = 0.0001$

$v_c(t) = 1 - e^{-10000t}$



charging capacitor

② by MATLAB ⇒ .m file

$$\frac{V_c}{V_1} = \frac{1}{1 + RCS}$$

Assume

⇓

$V_1 = u(t)$ ⇒ unit step

$R = 1\ \Omega$

$C = 0.0001$

```
R = 1;
C = 0.0001;
num = 1;
den = [R*C  1];
sys = tf(num, den);
t = 0 : 0.001 : 0.01;
Vc = step(sys, t);
plot(t, Vc)
```

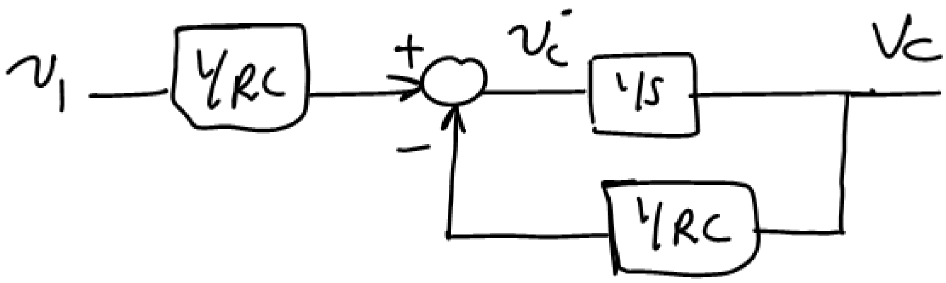
By SIMULINK

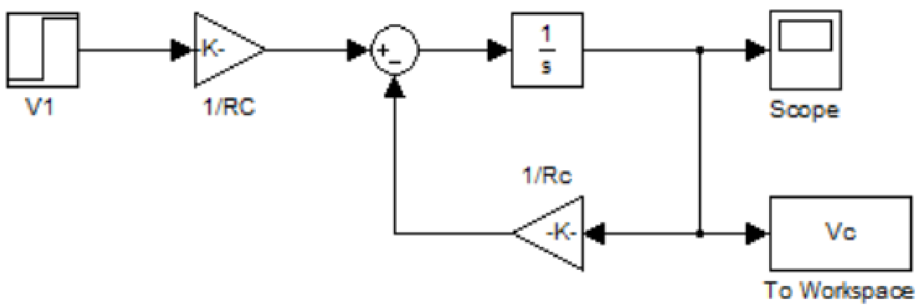
$$\frac{V_c}{V_1} = \frac{1}{1 + RCs}$$

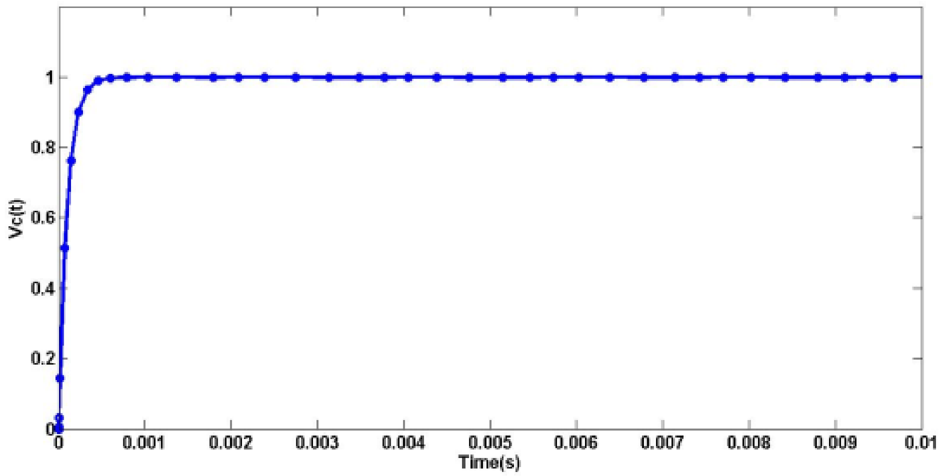
Taking inverse Laplace

$$v_c + RC\dot{v}_c = v_1$$

$$\frac{1}{RC}v_1 + \frac{1}{RC}v_c = \dot{v}_c$$

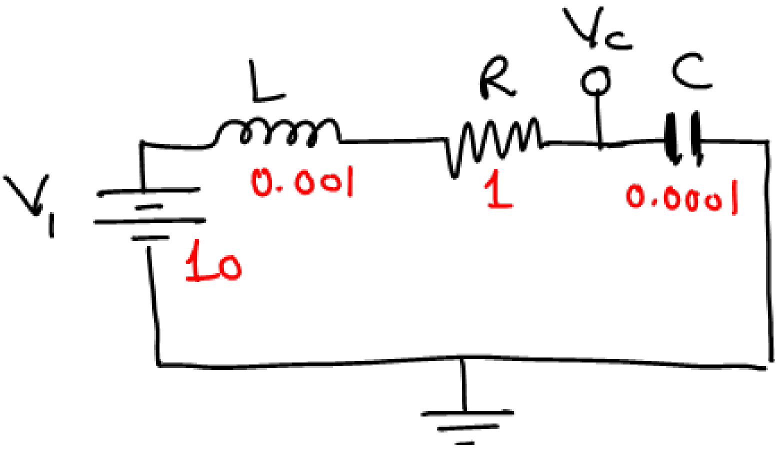






Example:

Use the SIMULINK to solve the following RLC circuit



$$v_1(t) = L\frac{di(t)}{dt} + i(t)\,R + \frac{1}{C}\int i(t)\,dt$$

Take Laplace

$$V_1(s) = LsI(s) + \ I(s)\,R + \frac{I(s)}{Cs}$$

$$V_1 = I\left(R + sL + \frac{1}{Cs}\right) = \frac{LCs^2 + RCs + 1}{Cs}$$

$$v_c(t) = \frac{1}{C}\int i(t)\, dt$$

$$V_c(s) = \frac{I(s)}{Cs}$$

$$\therefore \quad \frac{V_c}{V_1} = \frac{1}{LCs^2 + RCs + 1}$$

$$\frac{V_c}{V_1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

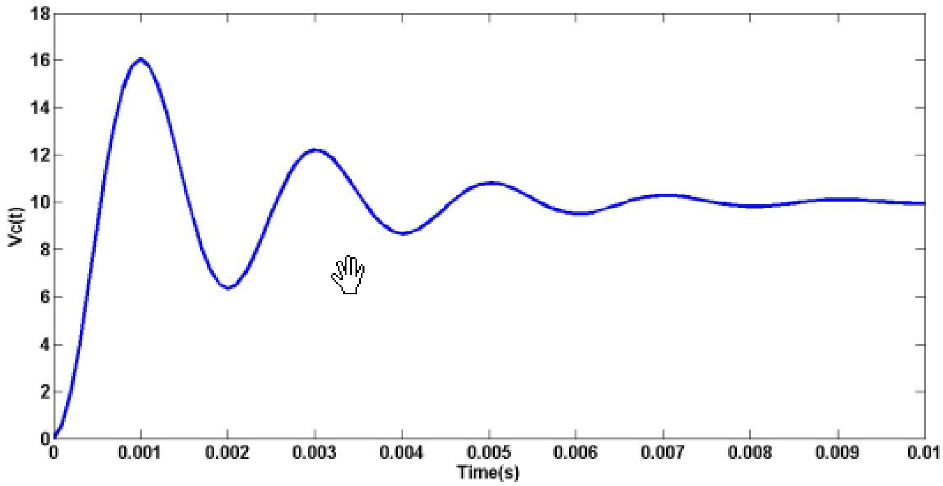
Assume $v_1(t) = 10\, u(t)$, $R = 1$, $C = 0.0001$, $L = 0.001$

Analytically

$$V_c = \frac{10^8}{s^2 + 1000s + 10^8}$$

$$= \frac{10}{s} - 10\frac{s + 500}{(s + 500)^2 + 9750000} + \frac{4000}{(s + 500)^2 + 9750000}$$

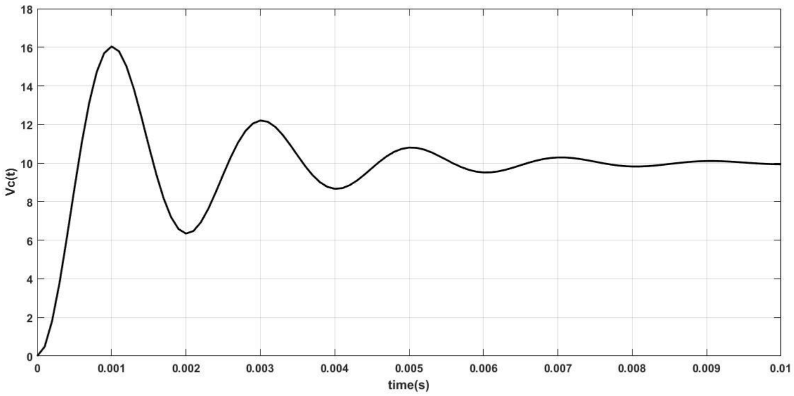
$$v_c(t) = 10 - 10e^{-500t}\cos\sqrt{9750000}t - \frac{4000}{\sqrt{9750000}}e^{-500t}\sin\sqrt{9750000}t$$



BY MATLAB // .m file

$$\frac{V_c}{V_1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

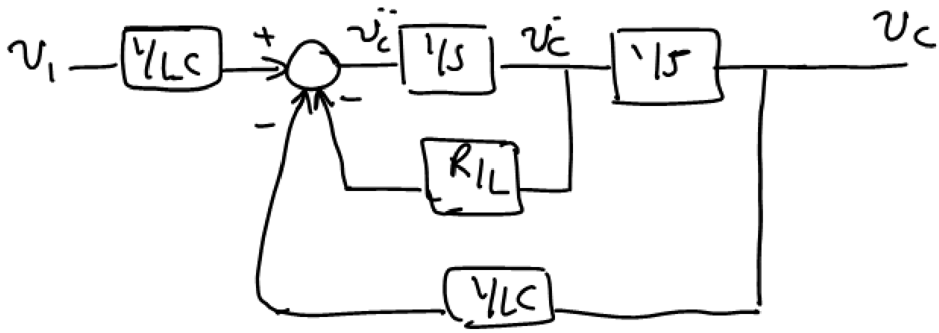
```
R =1;
L = 0.001;
C = 0.0001;
num = 1/(L*C);
den = [1 R/L 1/(L*C)];
sys = tf(num,den);
t = 0:0.0001:0.01;
Vc = step(10*sys,t);
plot(t,Vc)
```

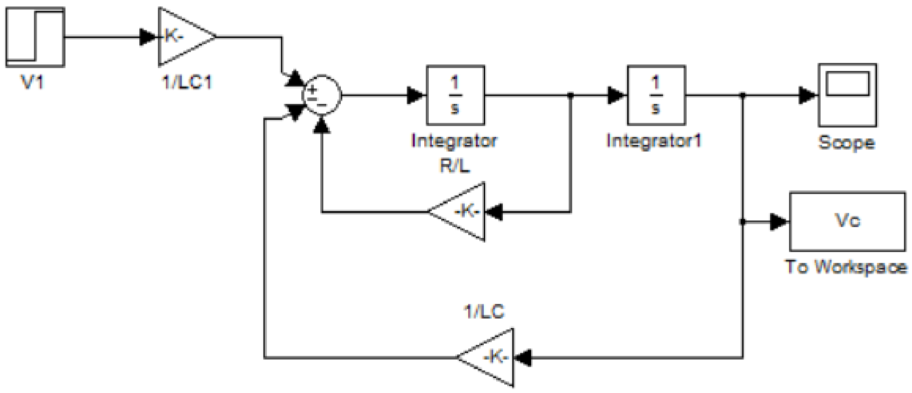


BY SIMULINK

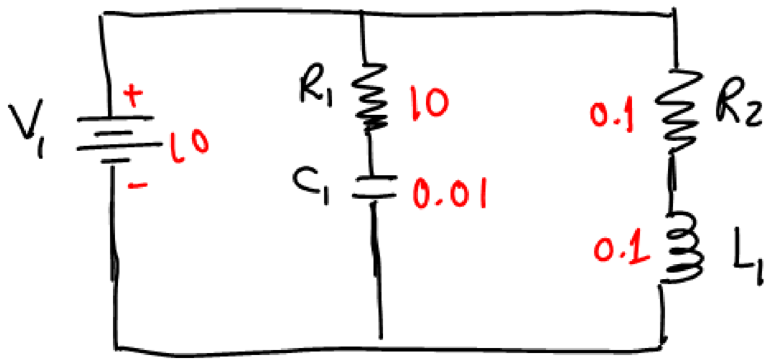
$$\frac{V_c}{V_1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\ddot{v}_c + \frac{R}{L}\dot{v}_c + \frac{1}{LC}v_c = \frac{1}{LC}v_1$$

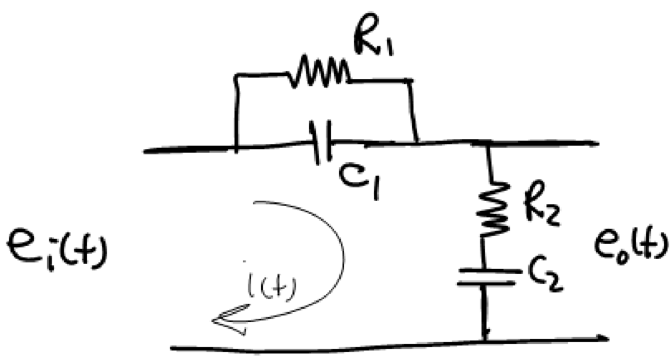




Problem:

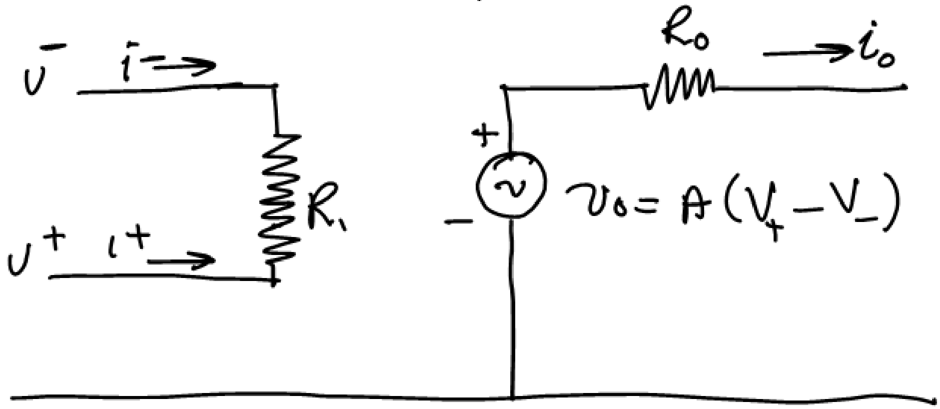


Problem:

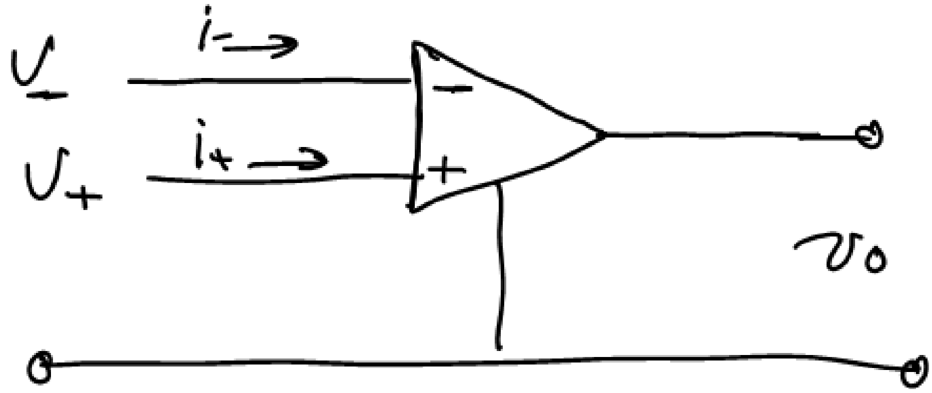


- Operational Amplifier

The simplified circuit of the Op-Amp is shown

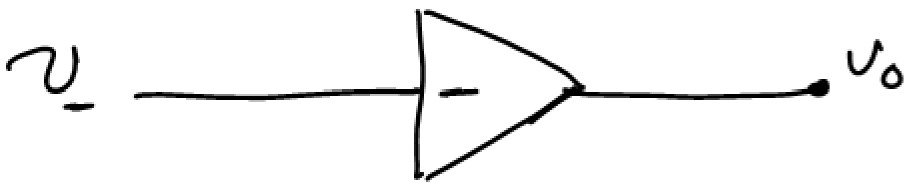


Schematic symbol



If the positive terminal is not shown, it is assumed to be connected to ground, $v^+ = 0$

And the reduced symbol below is used



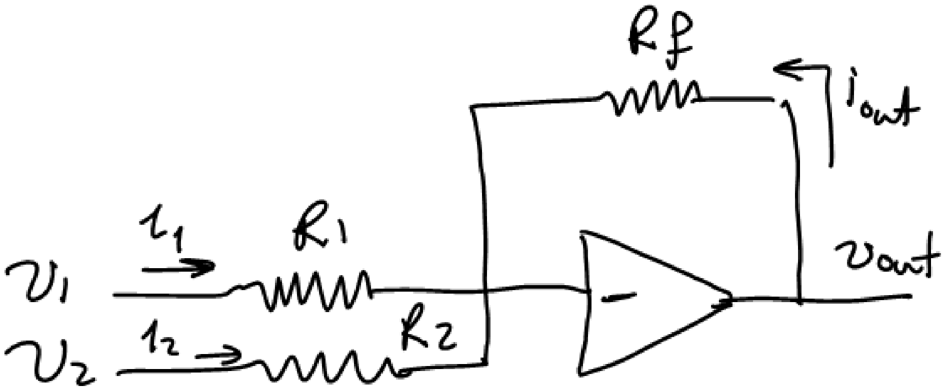
It is usually assumed that the OP-Amp is ideal with the values $R_{in} = \infty, R_{out} = 0, A = \infty$

$i^+ = i^- = 0$

$v^+ = v^-$

Example:

Find the equation and the transfer function of the circuit shown



$\because \quad v^+ = 0$

$\therefore \quad v^- = 0$

$$i_1 = \frac{v_1}{R_1}$$

$$i_2 = \frac{v_2}{R_2}$$

$$i_{out} = \frac{v_{out}}{R_f}$$

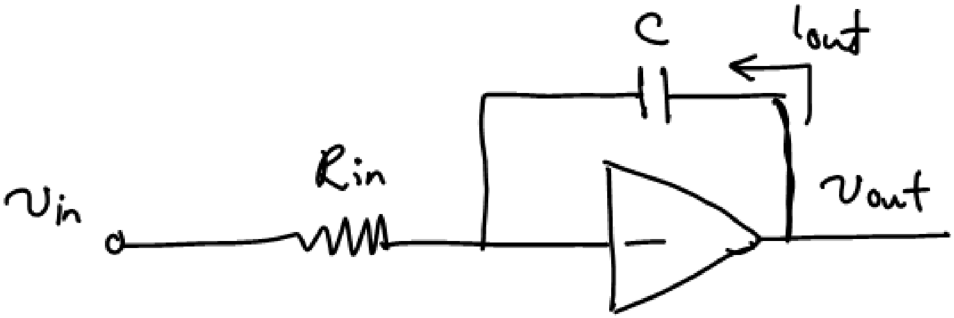
$$i_1 + i_2 + i_3 = 0$$

$$\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_{out}}{R_f} = 0$$

$$\therefore \quad v_{out} = -\left[\frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2\right]$$

The circuit O/P is a weighted sum of the input voltages with a sign change ➔ **summer circuit**

Example: Integrator



$$\because \quad v^+ = 0$$

$$\therefore \quad v^- = 0$$

$$\frac{v_{in}}{R_{in}} + C\frac{dv_{out}}{dt} = 0$$

$$\therefore \quad \frac{dv_{out}}{dt} = -\frac{v_{in}}{R_{in}C}$$

$$\frac{dv_{out}}{dt} = -\frac{1}{R_{in}C}\int_0^t v_{in}(\tau)\,d\tau + v_{out}(0)$$

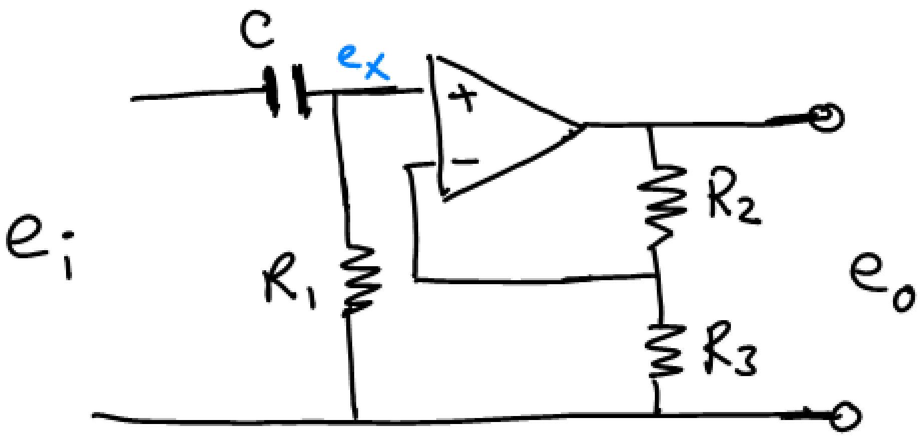
OR

$$sv_{out} = -\frac{1}{R_{in}C}v_{in}$$

$$\therefore \quad \frac{v_{out}}{v_{in}} = -\frac{1}{R_{in}C}$$

Example:

Find E_0/E_i



$$\frac{E_x}{E_i} = \frac{R_1}{R_1 + 1/sC}$$

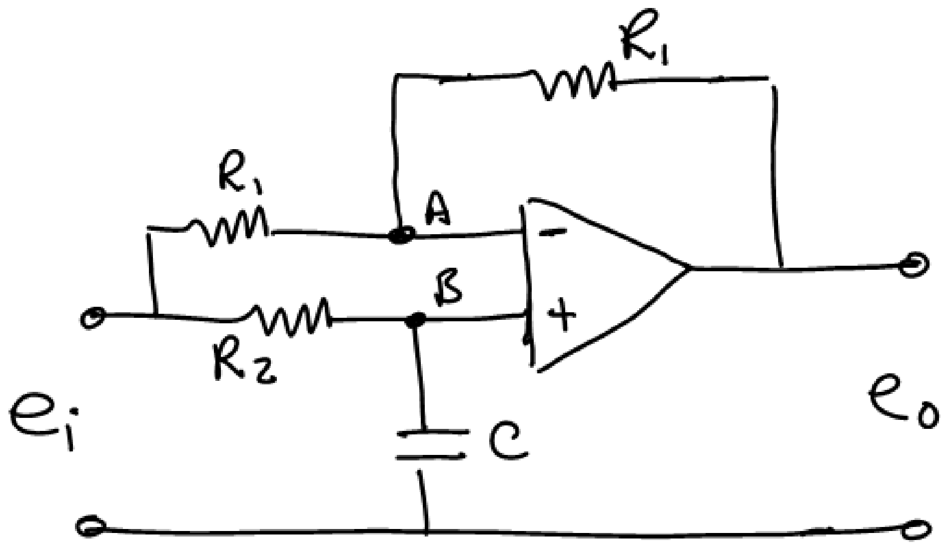
$$\frac{E_x}{E_o} = \frac{R_3}{R_2 + R_3}$$

$$\frac{E_o}{E_i} = \frac{R_1}{R_1 + 1/sC} \frac{R_2 + R_3}{R_3}$$

$$\frac{E_o}{E_i} = \frac{R_1(R_2 + R_3)}{R_3(R_1 + 1/sC)}$$

Problem:

Find E_0/E_i



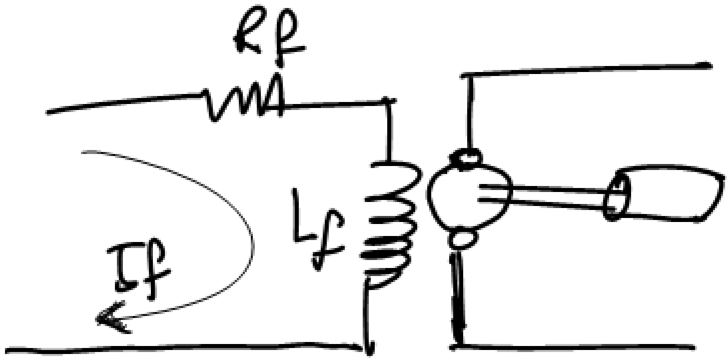
■ DC Motor

$$Torque \propto i_a i_f$$

Field control

i_a : constant

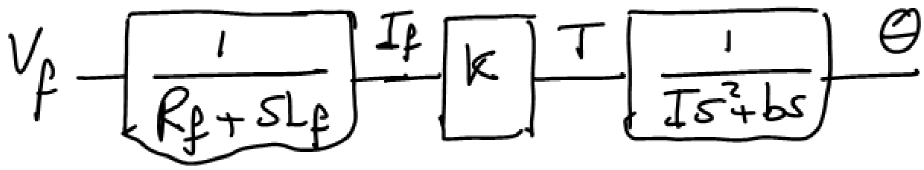
$Torque \propto i_f$



$V_f = I_f(R_f + sL_f)$

$T = kI_f$

$T = (Is^2 + bs)\theta$

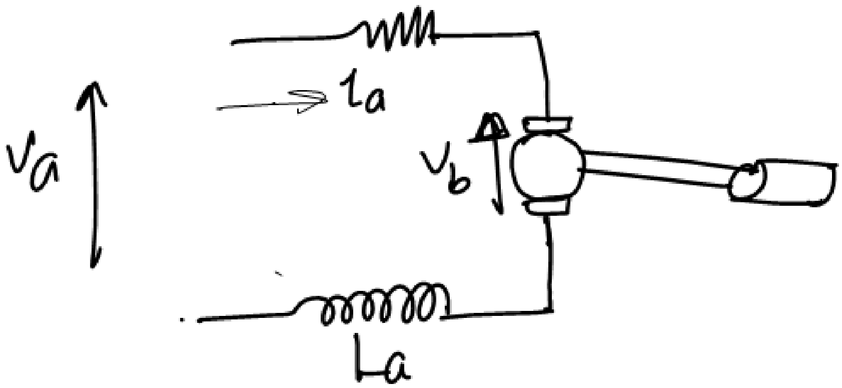


$$\frac{\theta}{V_f} = \frac{k}{(R_f + sL_f)(Is^2 + bs)}$$

Armature control

i_f : constant

$Torque \propto i_a$



$V_a - V_b = I_a(R + sL_a)$

$v_b \propto \dot{\theta}$

$V_b = k_b s\, \theta$

$T = kI_a$

$T = (Is^2 + bs)\theta$