Chapter 06

Time Response

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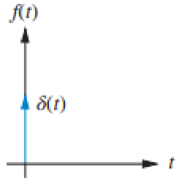
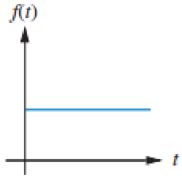
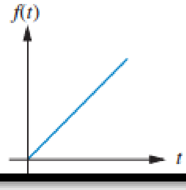
Introduction

- Once mathematical model has been obtained for a control system, a number of techniques can be applied to analyze the system's performance.
- The next step following block diagram reduction, is **analysis and design**. If you are interested only in the performance of an individual subsystem, you can skip the block diagram reduction and move immediately into analysis and design. In this phase, the engineer analyzes the system to see if the **response specifications and performance requirements can be met by simple adjustments of system parameters**. If specifications cannot be met, the designer then designs additional hardware "**Controller**" in order to affect a desired performance.
- One of the important analysis techniques is the time-domain analysis
- The manner in which a **dynamic system responds to an input, expressed as a function of time**, is called the **time response**
- Static response characteristic, such as steady-state errors and transient response characteristic for various input signals are covered

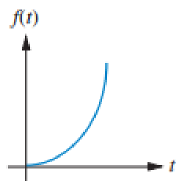
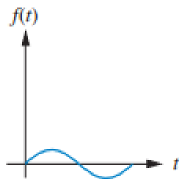
▪ Test signal

Test input signals are used, both analytically and during testing, to verify the design. It is neither necessarily practical nor illuminating to choose complicated input signals to analyze a system's performance. Thus, the **engineer usually selects standard test inputs**. These inputs are **impulses**, **steps**, **ramps**, **parabolas**, and **sinusoids**, as shown

Test waveforms used in control systems

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error

Which of these typical input signals to use for analyzing system characteristics may be **determined by the form of the input that the system will be subjected to most frequently under normal operation**. If the inputs to a control system are **gradually changing functions of time**, then a **ramp** function of time may be a good test signal. Similarly, if a system is subjected to **sudden disturbances**, a **step function** of time may be a good test signal; and for a system subjected to **shock inputs**, an **impulse** function may be best.

Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

An **impulse** is infinite at $t = 0$ and zero elsewhere. The area under the unit impulse is 1. An approximation of this type of waveform is used to place initial energy into a system so that the response due to that initial energy is only the **transient response of a system**. From this response the designer can derive a mathematical model of the system.

A **step input** represents a **constant command**, such as **position**, **velocity**, or **acceleration**. Typically, the step input command is of the **same form as the output**. For example, if the system's output is position, as it is for the antenna azimuth position control system, **the step input represents a desired position**, and the output represents the actual position. If the system's output is velocity, as is the spindle speed for a video disc player, the **step input represents a constant desired speed**, and the output represents the actual speed. The designer uses step inputs because both the transient response and the steady-state response are clearly visible and can be evaluated.

The **ramp input** represents a **linearly increasing command**. For example, if the system's output is position, the input ramp represents a linearly increasing position, such as that found when tracking a satellite moving across the sky at constant speed. If the system's output is velocity, the input ramp represents a linearly increasing velocity. **The response to an input ramp test signal yields additional information about the steady-state error**. The previous discussion can be extended to **parabolic inputs**, which are also used to evaluate a system's steady-state error.

Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory. The use of such test signals **enables one to compare the performance of many systems on the same basis**

The control systems engineer must take into consideration other characteristics about feedback control systems. For example, **control system behavior is altered by fluctuations in component values or system parameters**. These variations can be **caused by temperature, pressure**, or other environmental changes. Systems must be built so that **expected fluctuations do not degrade performance beyond specified bounds**. A **sensitivity analysis** can yield the percentage of change in a specification as a function of a change in a system parameter. One of the **designer's goals, then, is to build a system with minimum sensitivity over an expected range of environmental changes**

- the designer is concerned about **transient response**, **steady-state error**, **stability**, and **sensitivity**

Transient Response of a first-order system

Elements are said to be first order when the relationship between the input and the output **involves the rate at which the output changes**

$$\boldsymbol{a_1 \frac{dy}{dt} + a_0 y = b_0 x}$$

Where

$\boldsymbol{y}$: output ➔ measured quantity

$\boldsymbol{x}$: input quantity

$\boldsymbol{t}$: time

- We no longer have a simple relationship between the input and the output but a relationship which involves time
- the transducer that contains storage element cannot respond instantaneously to change an input.
    - The mercury in glass thermometer
    - Velocity of a true falling mass
    - Air pressure build-up in bellow
- To illustrate this, consider a **thermometer at temperature $\boldsymbol{T_0}$ inserted into a liquid at a temperature $\boldsymbol{T_1}$**. We can thus think of the thermometer being subject to step input (abruptly changes from $\boldsymbol{T_0}$ **to** $\boldsymbol{T_1}$). Thus, we have a measurement system, the thermometer, which has a step input and an output which changes from T_0 to T_1 over some time.

  The question we are concerned with ➔ is how the output, i.e. the reading of the thermometer T, varies with time
- The **solution to the above differential equation** indicates how the output T changes with time t when there is a step input

$$\boldsymbol{T(t) = T_1 + (T_0 - T_1)e^{-t/\tau}}$$

Where:

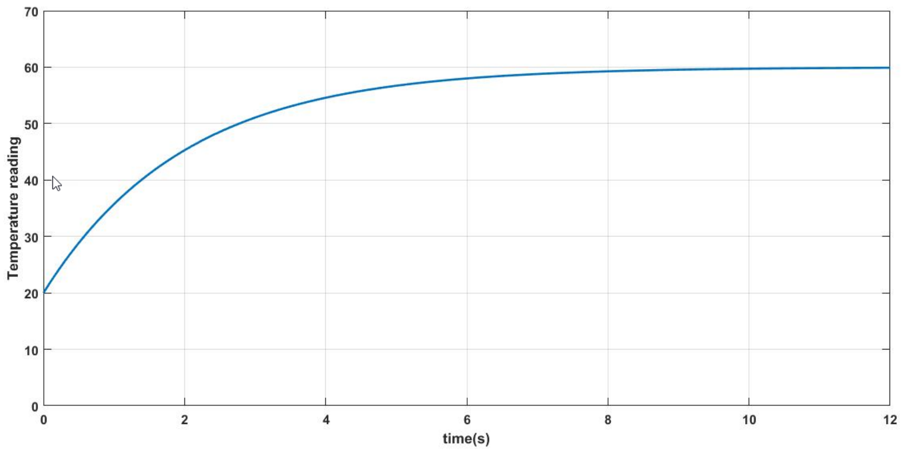
τ : time constant

Time constant is the time the system takes to reach 63.2% of its final value (see Table above).

T_1 : steady-state value – after sufficient time has elapsed for all transients to die away

Plot the response of the thermometer "First-order system" at temperature T_0 inserted into a liquid at a temperature T_1

```
T0 = 20;
T1 = 60;
t = 0:0.01:20;
tau = 2;
T2 = T1+(T0-T1)*exp(-t/tau);
plot(t,T2)
```

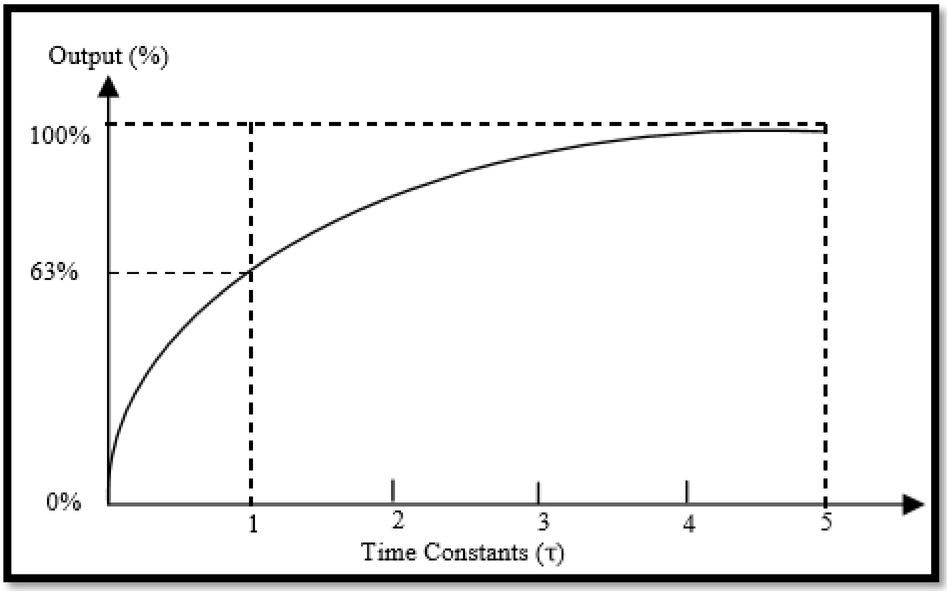


- The system response is called a “**first-order lag**” because the output **lags** behind the input, and the **differential equation** for the system shown in Figure below is a **linear first-order differential equation**.
- The **first-order lag** is the most common type of time element encountered in process control

✍ **Note:**

- If the thermometer is initially at zero temperature, i.e. $T_0 = 0$ ℃

$$\boldsymbol{T(t) = T_1(1 - e^{-t/\tau})}$$



Response of first order system to step input

- After a time equal to one time constant the output reached about 63% of the way to the steady-state temperature, after a time equal to two time constant, the output reached about 86% of the way, after three-time constants about 95% and after about four time constants, it is virtual equal to the steady-state value
- The error at any time is the difference between what the thermometer is indicating and what the temperature actually is. Thus:

$$\text{Error} = T - T_1$$

And so:

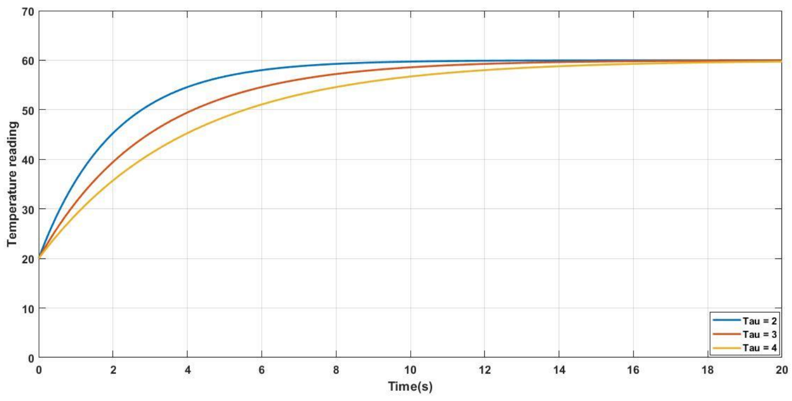
$$\text{Error} = (\boldsymbol{T_0} - \boldsymbol{T_1})\boldsymbol{e^{-t/\tau}}$$

- To illustrate the above, consider a thermometer which is indicating temperature of $20\,℃$ when it is suddenly immersed in a liquid at a temperature of $60\,℃$. If the thermometer has a time constant of $5\,s$ then the temperature T of the thermometer varies with time according to the equation

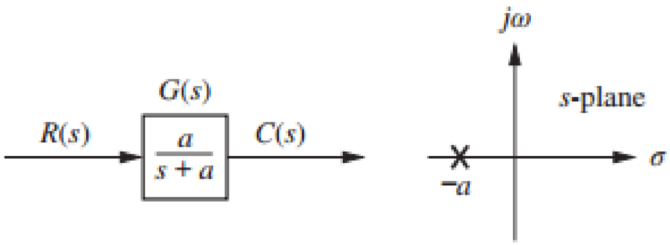
$$\boldsymbol{T(t) = T_1 + (T_0 - T_1)e^{-\frac{t}{\tau}} = 60 - 40\, e^{-t/\tau}}$$

After 5 s the thermometer reading will have reached about 63% of the way to the steady-state value, after 10 s about 86%, after 15 s, about 95% and after 20 s it is virtually at the steady state value. Thus after 5 s, the reading is 45.3 ℃, after 10 s it is 54.6 ℃, after 15 s it is 58.0 ℃

- The dynamic shape of their response to a step input is a function of their time constant. This system time constant, designated by the Greek letter **τ** (**tau**), is meaningful both in a physical and a mathematical sense. Physically, **it determines the shape of the response of a process or system to a step input**. Mathematically, it predicts, at any instant, the future time period that is required to obtain 63.2 percent of the change remaining. The response curve in Figure below illustrates the concept of a time constant by showing the response of a simple first-order system to a step input



A first-order system without zeros can be described by the transfer function shown. If the input is a unit step. Find the output of the system



$$C(s) = R(s) * G(s)$$

$$C(s) = \frac{a}{s+a} * R(s)$$

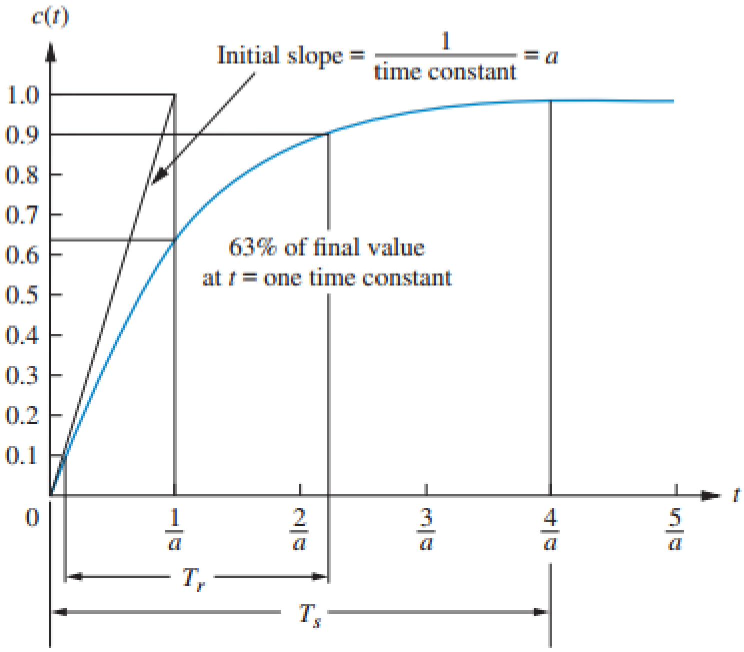
$$C(s) = \frac{a}{s+a} * \frac{1}{s} = \frac{1}{s} - \frac{1}{s+a}$$

$$C(t) = 1 - e^{-at}$$

Let us examine the significance of parameter a, the only parameter needed to describe the transient response. When $t = 1/a$,

$$C(t)|_{t=1/a} = 1 - e^{-1} = 1 - 0.37 = 0.63$$

We call $1/a$ the time constant of the response



The **reciprocal of the time constant** has the units (1/seconds), or frequency. Thus, we can call the parameter a the **exponential frequency**. Since the derivative of $\boldsymbol{e^{-at}}$ is $\boldsymbol{-a}$ when $t = 0$, a is the initial rate of change of the exponential at $t = 0$. **Thus, the time constant can be considered a transient response specification for a first-order system**, since it is related to the speed at which the system responds to a step input. The time constant can also be evaluated from the pole plot. Since the pole of the transfer function is at $\boldsymbol{a}$, we can say the pole is located at the *reciprocal* of the time constant, and the farther the pole from the imaginary axis, the faster the transient response.

- **Rise time:**

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

Rise time is found by solving $C(t) = 1 - e^{-at}$ for the difference in time at $c(t) = 0.9$ and $c(t) = 0.1$, Hence

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

- **Settling time:**

Settling time is defined as the time for the response to reach, and stay within, 2% of its final value. Letting $c(t) = 0.98$ in $C(t) = 1 - e^{-at}$ and solving for time, t, we find the settling time to be

$$T_s = \frac{4}{a}$$

- **First-order transfer functions via Testing**

Often it is not possible or practical to obtain a system's transfer function analytically. Perhaps the system is closed, and the component parts are not easily identifiable. Since the transfer function is a representation of the system from input to output, the system's step response can lead to a representation even though the inner construction is not known.

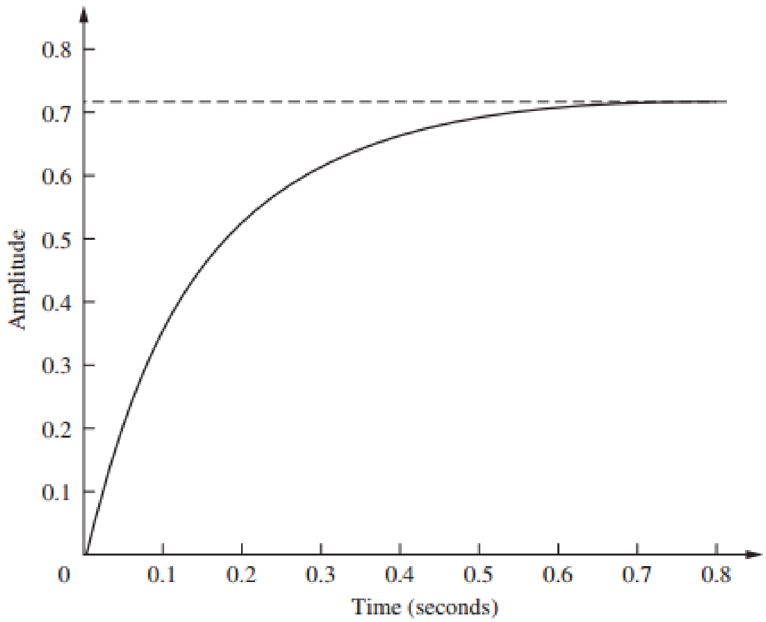
With a step input, we can **measure the time constant** and the **steady-state value**, from which the **transfer function can be calculated**.

Consider a simple first-order system, $G(s) = K/(s + a)$ whose step response is

$$C(s) = \frac{K}{s(s + a)} = \frac{K/a}{s} - \frac{K/a}{s + a}$$

If we can identify K and a from laboratory testing, we can obtain the transfer function of the system.

For example, assume the unit step response given in Figure below. We determine that it has the **first-order** characteristics we have seen thus far, such as **no overshoot** and **nonzero initial slope**.



From the response, we **measure** the **time constant**, that is, the time for the amplitude to reach 63% of its final value. Since the **final value is about $\mathbf{0.72}$**, the time constant is evaluated where the curve reaches $\mathbf{0{:}63 \times 0{:}72 = 0.45}$, or about $\mathbf{0.13}$ second. Hence, $\boldsymbol{a = \frac{1}{0.13} = 7.7}$

To find K, we realize from Eq.

$$\boldsymbol{C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{s+a}}$$

that the forced response reaches a steady-state value of $\boldsymbol{K/a = 0.72}$. Substituting the value of $\boldsymbol{a}$, we find $\boldsymbol{K = 5.54}$. Thus, the transfer function for the system is $\boldsymbol{G(s) = 5.54/(s+7.7)}$

A system has a transfer function, $\boldsymbol{G(s) = \frac{50}{s+50}}$. Find the time constant, τ, settling time, T_s , and rise time, T_r.

$\tau = 0.02\ s$ $\qquad T_s = 0.08\ s$ $\qquad T_r = 0.044\ s$

- **Response of first order system to step input**

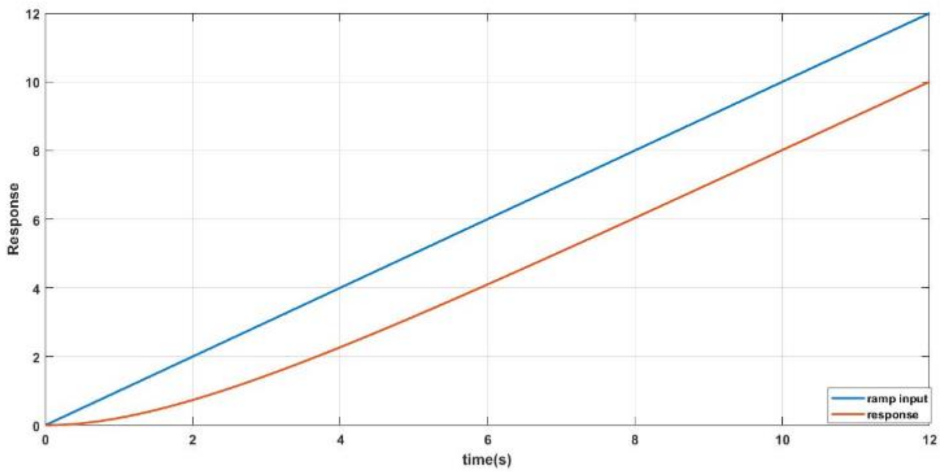
Since the Laplace transform of the unit-ramp function is $1/s^2$, we get the output of the 1st-order system as

$$C(s) = \frac{1}{s^2} * \frac{1}{\tau s + 1}$$

$$C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

$$c(t) = t - \tau + \tau e^{-t/\tau}$$

```
T0 = 0;
a = 1;    % slope
t = 0:0.01:12;
%%
T1 = T0+a*t;
tau = 2;
T2 = T1-a*tau+ a*tau*exp(-t/tau);
plot(t,T1,t,T2)
```



The error signal is $e(t)$ is then

$e(t) = r(t) - c(t)$

$e(t) = \tau(1 - e^{-t/\tau})$

As t approaches infinity, $e^{-t/\tau}$ approaches zero, and thus the error signal $e(t)$ approaches τ or

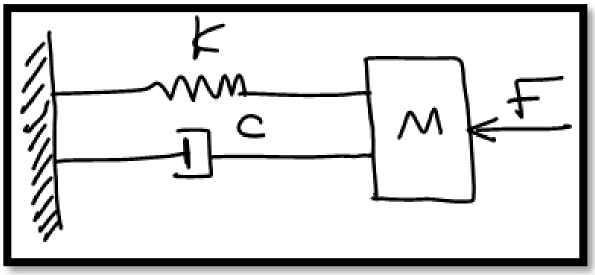
$$e(\infty) = \tau$$

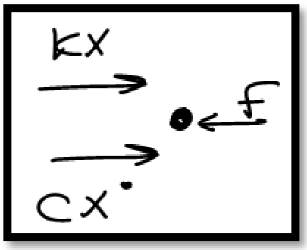
The smaller the time constant τ, the smaller the steady-state error in following the ramp input

■ Second-order system response

$$a_2\frac{d^2y}{dt^2}+a_1\frac{dy}{dt}+a_0y=b_0x$$

Compared to the simplicity of a first-order system, a **second-order system exhibits a wide range of responses that must be analyzed and described**. Whereas varying a first-order system's parameter simply changes the speed of the response, **changes in the parameters of a second-order system can change the form of the response**. For example, a second-order system can display characteristics much like a first-order system, or, depending on component values, **display damped** or **pure oscillations** for its transient response





$$\sum F=m\ddot{x}$$

$$F-c\dot{x}-kx=\ m\ddot{x}$$

$$(ms^2+cs+k)X=F$$

$$\frac{X}{F} = \frac{1}{(ms^2 + cs + k)}$$

In the absence of damping ➔ $c = 0$

$$\frac{X}{F} = \frac{1}{ms^2 + k} = \frac{1/m}{s^2 + k/m}$$

$$\frac{X}{F} = \frac{1/m}{s^2+k/m}$$

there will be a term in t-domain that contains cos $\sqrt{\frac{k}{m}}t$

$$\therefore \quad \boldsymbol{\omega_n = \sqrt{\frac{k}{m}}}$$

mass m on the end of a spring freely oscillates with a natural frequency $\boldsymbol{\omega_n = \sqrt{\frac{k}{m}}}$

$$\because \quad \frac{X}{F} = \frac{1}{ms^2 + cs + k}$$

Divide by m

$$\therefore \quad \boldsymbol{\frac{X}{F} = \frac{1/m}{s^2 + \frac{c}{m}s + k/m}}$$

$$\because \quad \boldsymbol{\omega_n = \sqrt{\frac{k}{m}}} \qquad\qquad \therefore \quad \boldsymbol{{\omega_n}^2 = k/m}$$

Assume another term called $\zeta = \frac{c}{2\sqrt{mk}}$

Multiply $\boldsymbol{\omega_n}$ by ζ equation

$$\therefore \quad \zeta\boldsymbol{\omega_n} = \frac{c}{2m}$$

$$\therefore \quad 2\zeta\omega_n = \frac{c}{m}$$

$$\because \quad \frac{\boldsymbol{X}}{\boldsymbol{F}} = \frac{\boldsymbol{1/m}}{\boldsymbol{s^2 + \frac{c}{m}s + k/m}}$$

$$\boxed{\therefore \quad \frac{\boldsymbol{X}}{\boldsymbol{F}} = \frac{\boldsymbol{1/m}}{\boldsymbol{s^2 + 2\zeta\omega_n s + \omega_n{}^2}}}$$

<u>If you are interested in step response of an instrument with second-order system</u>

Assume $f(t)$ is a step input with an amplitude F ➔ $f(t) = F$

$$\therefore \quad F(s) = \frac{F}{s}$$

$$\therefore \quad \boldsymbol{X} = \frac{\boldsymbol{F/m}}{\boldsymbol{s(s^2 + 2\zeta\omega_n s + \omega_n{}^2)}}$$

use partial fraction to get

$$\boldsymbol{X(s)} = \frac{\boldsymbol{F}}{\boldsymbol{k}}\left[\frac{\boldsymbol{1}}{\boldsymbol{s}} - \frac{\boldsymbol{s + 2\zeta\omega_n}}{\boldsymbol{s^2 + 2\zeta\omega_n s + \omega_n{}^2}}\right]$$

The time-domain solution to this equation depends on the value of $\boldsymbol{\zeta}$, so we have:

Underdamped case — $0 < \zeta < 1$

Report

Sketch the transient response of a 2nd order system to a unit step for different values of ζ in the range of $0 < \zeta < 1$ — hint: draw with respect to $\boldsymbol{\omega_n t}$

critically damped case — $\zeta = 1$

Report

Sketch the transient response of a 2nd order system to a unit step for $\zeta = 1$ — hint: draw with respect to $\boldsymbol{\omega_n t}$

Overdamped case — $\zeta > 1$

Report

Sketch the transient response of a 2nd order system to a unit step for different values of ζ in the range of $\zeta > 1$ — hint: draw with respect to $\boldsymbol{\omega_n t}$

Characteristic equation:

The standard form of the second-order system

$$T.F = \frac{1/m}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

ω_n: undamped natural frequency (rad/s) and

ζ : Damping ratio

ω_d: damped natural frequency $\omega_n\sqrt{1-\zeta^2}$

C/Cs **equation**: $s^2 + 2\zeta\omega_n s + {\omega_n}^2$

This polynomial in s is called the **Characteristic Equation** and its ***roots will determine the system transient response.***

The dynamic behavior of the second-order system can then be described in terms of two parameters ζ and ω_n.

- If $0 < \zeta < 1$, the **closed-loop poles** are complex conjugates and lie in the **left-half s plane**. The system is then called **underdamped**, and the transient response is **oscillatory**.
- If $\zeta = 0$, the transient response does not die out.
- If $\zeta = 1$, the system is called **critically damped**.

- **Overdamped** systems correspond to $\boldsymbol{\zeta > 1}$.

Recall again that, the transfer function of the 2nd-order system is

$$\boldsymbol{T.F = \frac{1/m}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}}$$

And its roots can be obtained as

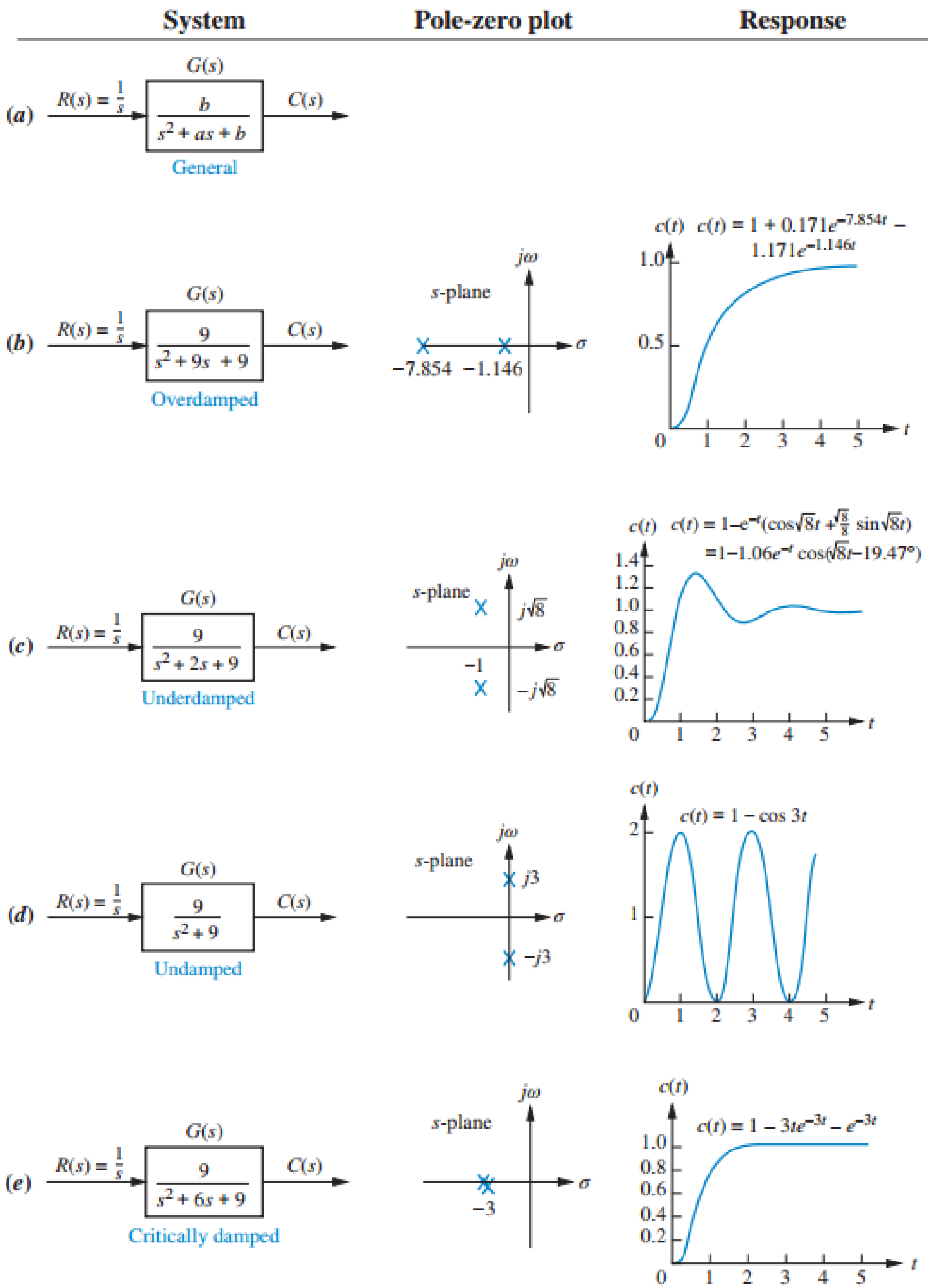
$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\boldsymbol{s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$

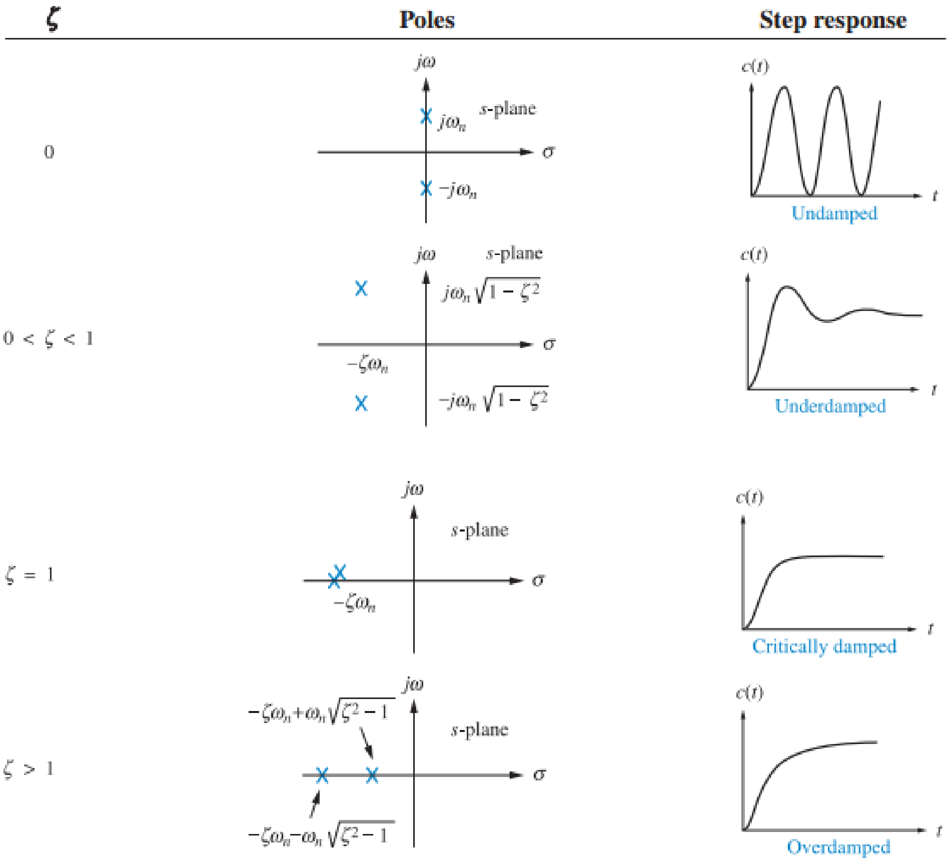
Depending on the values of the term $(b^2 - 4ac)$, called the ***discriminant***, may be positive, zero or negative which will make the roots real and unequal, real and equal or complex. This gives rise to the three different types of transient response described in Table below.

Transient behaviour of a second-order system

Discriminant	*Roots*	*Transient response type*
$b^2 > 4ac$	s_1 and s_2 real and unequal (−ve)	Overdamped Transient Response
$b^2 = 4ac$	s_1 and s_2 real and equal (−ve)	Critically Damped Transient Response
$b^2 < 4ac$	s_1 and s_2 complex conjugate of the form: $s_1, s_2 = -\sigma \pm j\omega$	Underdamped Transient Response

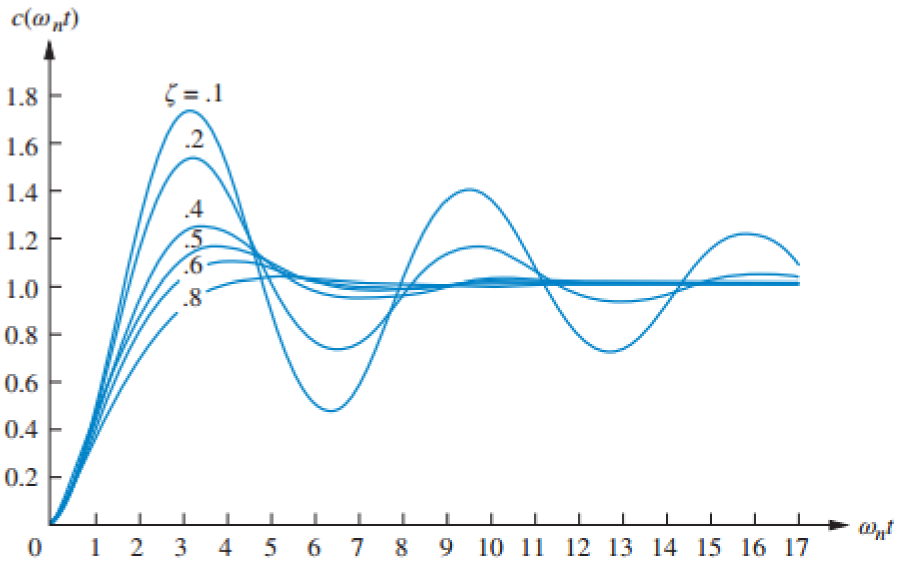


$C(s) = R(s)G(s)$, where $R(s) = 1/s$ followed by a partial-fraction expansion and the inverse Laplace transform.



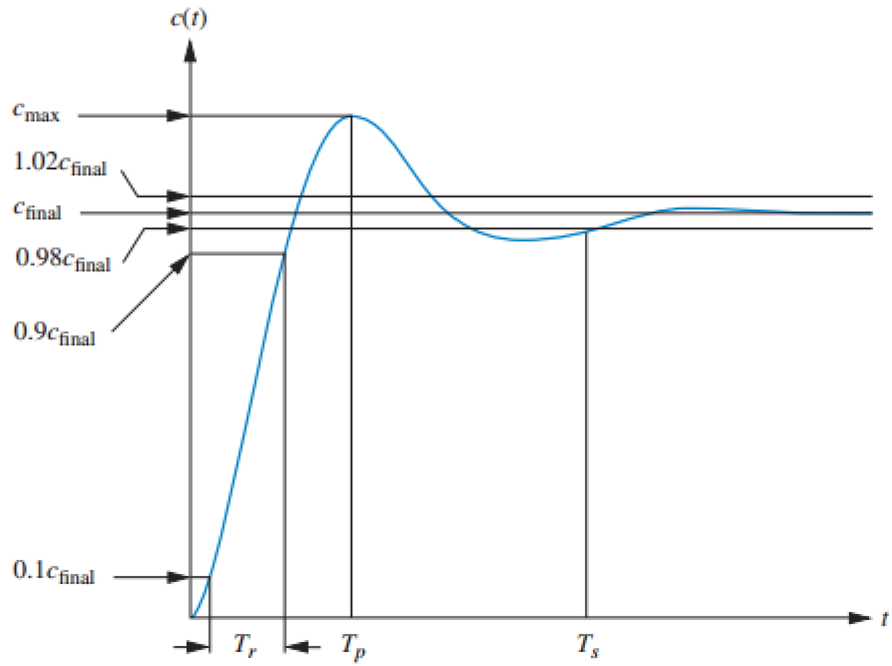
- Underdamped Second-Order Systems

let us analyze the step response of an *underdamped* second-order system. Not only will this response be found in terms of ζ and ω_n , but more specifications inherent to the underdamped case will be defined



We now see the relationship between the value of ζ and the type of response obtained: The lower the value of ζ, the more oscillatory the response. The natural frequency ω_n is a time-axis scale factor and does not affect the nature of the response other than to scale it in time.

We have defined two parameters associated with second-order systems, ζ and ω_n. Other parameters associated with the underdamped response are **rise time**, **peak time**, **percent overshoot**, and **settling time**. These specifications are defined as follows



Second-order underdamped response specifications

Rise time, T_r: The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

Peak time, T_p : The time required to reach the first, or maximum, peak.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Percent overshoot, $\%OS$: The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

$$\%OS = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Settling time, T_s: The time required for the transient's damped oscillations to reach and stay within 2% of the steady-state value

$$T_s = \frac{4}{\zeta \omega_n}$$

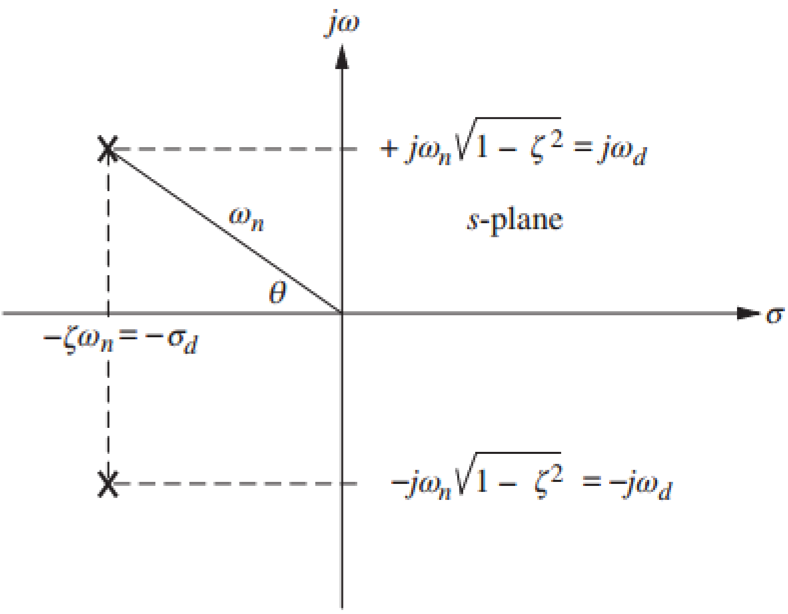
Notice that the definitions for settling time and rise time are basically the same as the definitions for the first-order response.

All definitions are also valid for systems of order higher than 2, although analytical expressions for these parameters cannot be found unless the response of the higher-order system can be approximated as a second-order system

Rise time, **peak time**, and **settling time** yield information about the speed of the transient response. This information can help a designer determine if the speed and the nature of the response do or do not degrade the performance of the system. For example, the speed of an entire computer system depends on the time it takes for a hard drive head to reach steady state and read data; passenger comfort depends in part on the suspension system of a car and the number of oscillations it goes through after hitting a bump.

Note:

We now have expressions that relate peak time, percent overshoot, and settling time to the natural frequency and the damping ratio. Now let us relate these quantities to the location of the poles that generate these characteristics.



Pole plot for an underdamped second-order system

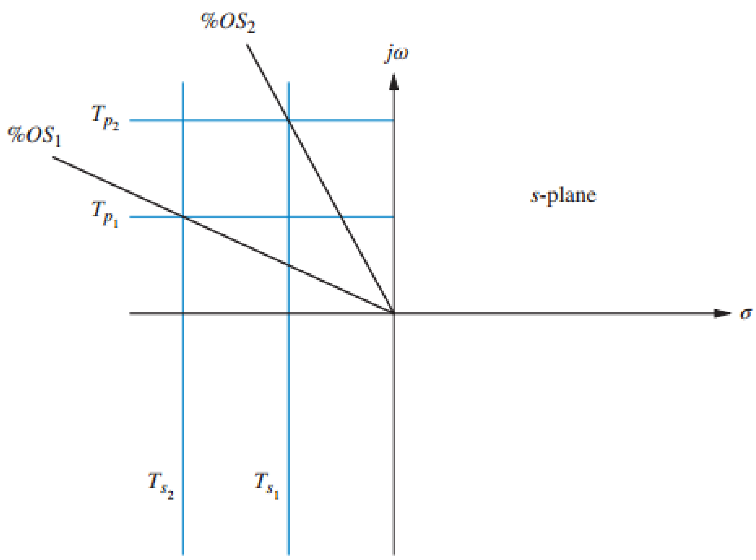
$$T_p = \frac{\pi}{\omega_d}$$

$$T_s = \frac{\pi}{\sigma_d}$$

$$\zeta = \cos\theta$$

Design criteria:

- Equation $T_p = \frac{\pi}{\omega_d}$ shows that T_p is inversely proportional to the imaginary part of the pole. Since horizontal lines on the s-plane are lines of constant imaginary value, they are also lines of constant peak time.
- Similarly, Eq. $T_s = \frac{\pi}{\sigma_d}$ tells us that settling time is inversely proportional to the real part of the pole. Since vertical lines on the s-plane are lines of constant real value, they are also lines of constant settling time.
- Finally, since $\zeta = \cos\theta$, radial lines are lines of constant ζ. Since percent overshoot is only a function of ζ, radial lines are thus lines of constant percent overshoot, $\%OS$. These concepts are depicted in Figure below, where lines of constant T_p, T_s, and $\%OS$ are labeled on the s-plane.



Poles, Zeros, and System Response

- The output response of a system is the sum of two responses: the forced response and the natural response.
- The **forced response** is also called the **steady-state response** or **particular solution**. The **natural response** is also called the **homogeneous solution**
- Although many techniques, such as **solving a differential equation or taking the inverse Laplace transform**, enable us to evaluate this output response, these techniques are laborious and time-consuming. Productivity is aided by analysis and design techniques that **yield results in a minimum of time**. If the technique is so rapid that we feel we derive the **desired result by inspection**, we sometimes use the attribute qualitative to describe the method. The **use of poles and zeros and their relationship to the time response of a system is such a technique**. Learning this relationship gives us a qualitative “handle” on problems. The concept of **poles and zeros**, fundamental to the analysis and design of control

- **Poles and zeros**

The closed loop transfer function $G(s)$ of a system can in general be represented by

$$G(s) = \frac{K(s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + \cdots + a_1 s + a_0)}{s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}$$

If the roots of the denominator and numerator are established then

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{s^n(s+p_1)(s+p_2)\ldots(s+p_n)}$$

The values of s for which the numerator of the system transfer function is equal to zero are called **zeros** of the system

When the denominator of the system transfer function is set equal to zero, then values of $\boldsymbol{s}$ obtained, then values of $\boldsymbol{s}$ obtained are called the **poles**

Therefore, the roots of the numerator $-\boldsymbol{z_1}, -\boldsymbol{z_2}, \ldots, -\boldsymbol{z_m}$ are called **zeros**

> ✍ **Note:** zeros are also any root of the numerator of the transfer function that are common to roots of the denominator.
>
> if a factor of the numerator can be canceled by the same

factor in the denominator, the root of this factor no longer causes the transfer function to become zero. In control systems, we **often refer to the root of the canceled factor in the numerator as a zero even though the transfer function will not be zero** at this value.

And the roots of the denominators $-\boldsymbol{p_1}, -\boldsymbol{p_2}, \dots, -\boldsymbol{p_n}$ are called **poles**

Poles are the values of the $\boldsymbol{s}$ for which the transfer function is infinite

If the denominator is $(\boldsymbol{s}+\boldsymbol{5})$, then the transfer function is infinite when $\boldsymbol{s} = -\boldsymbol{5}$. Hence, $\boldsymbol{s} = -\boldsymbol{5}$ is the pole

✍ **Note:** any roots of the denominator of the transfer function that are common to roots of the numerator. if a factor of the denominator can be canceled by the same factor in the numerator, the root of this factor no longer causes the transfer function to become infinite.

In control systems, we **often refer to the root of the canceled factor in the denominator as a pole** even though the transfer function will not be infinite at this value.

Zeros and **poles** can be **real** or **complex quantities**

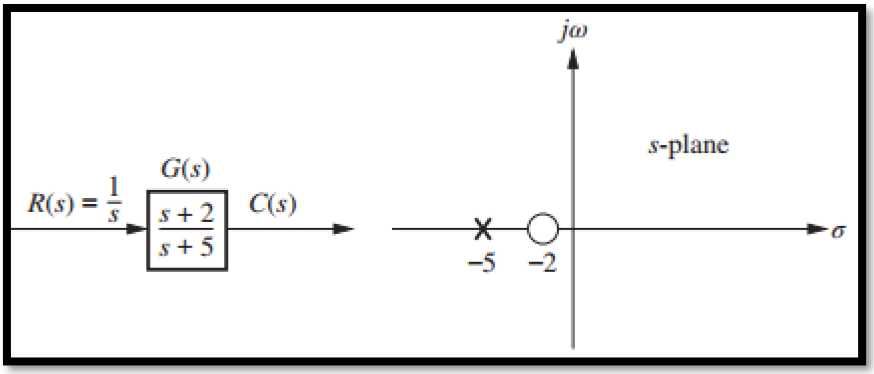
- **S-plane plots**

The Laplace variable s is a complex number; the values of s can be used to construct a plot. The **s-plane** is used to graphically represent the location of the poles and zeros of the transfer function

Given the transfer function

$$G(s) = \frac{(s+2)}{(s+5)}$$

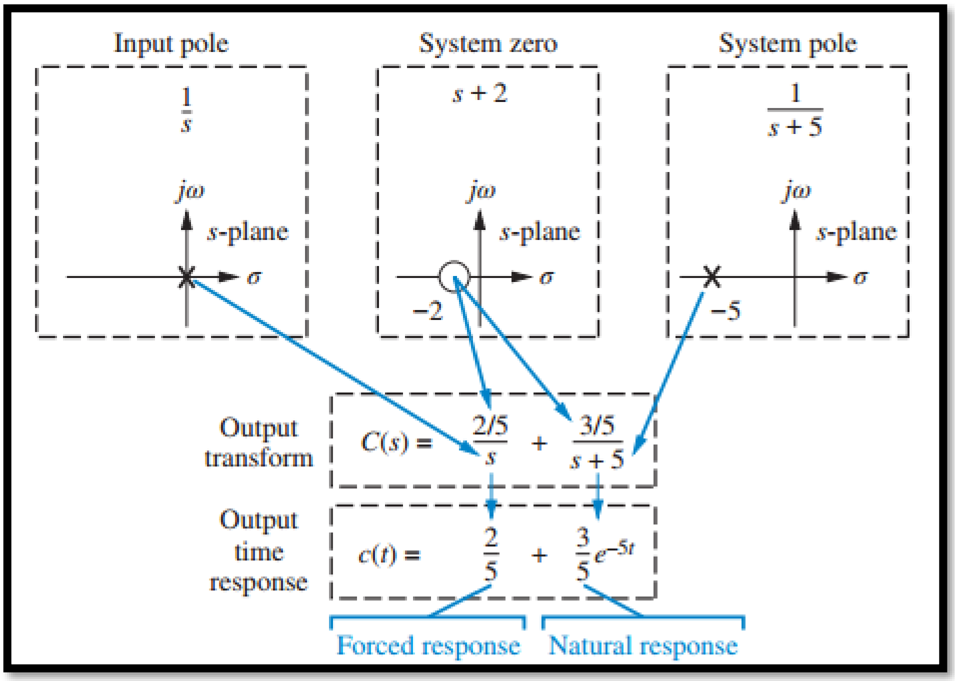
Pole exist at $s = -5$ and zero exists at $s = -2$



To show the properties of the poles and zeros, let us find the unit step response of the system. Multiplying the transfer function by a step function yields

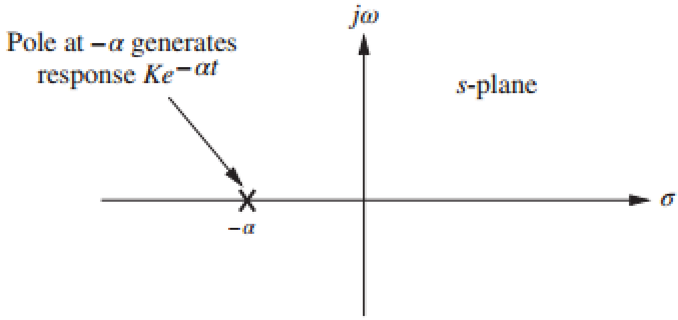
$$c(s) = \frac{(s+2)}{s(s+5)} = \frac{0.4}{s} + \frac{0.6}{s+5}$$

$$c(t) = 0.4 + 0.6e^{-5t}$$



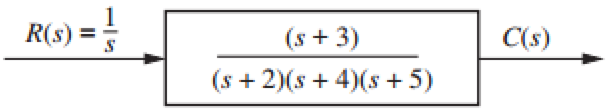
From the development summarized in Figure above, we draw the following conclusions:

- A **pole of the input function** generates the form of the **forced response** (that is, the pole at the origin generated a step function at the output).
- A pole of the transfer function generates the form of the natural response (that is, the pole at -5 generated e^{-5t}).
- A **pole on the real axis** generates an **exponential response** of the form $\boldsymbol{e^{-\alpha t}}$, where $\boldsymbol{\alpha}$ is the pole location on the real axis. **Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero** (again, the pole at 5 generated $\boldsymbol{e^{-5t}}$).



- The zeros and poles generate the amplitudes for both the forced and natural responses

Given the system below,

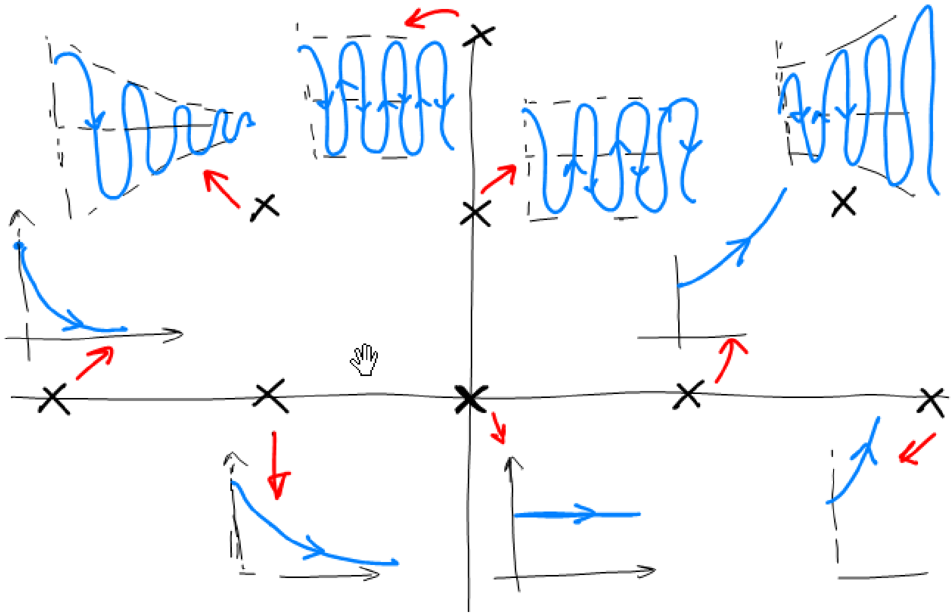


write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

$$c(s) = \frac{(s+3)}{s(s+2)(s+4)(s+5)} = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{s+4} + \frac{k_4}{s+5}$$

$$c(t) = k_1 + k_2 e^{-2t} + k_3 e^{-4t} + k_4 e^{-5t}$$

- Poles indicates response



Problem

1. For each of the following transfer functions, write, by inspection, the general form of the step response:
   a. $G(s) = \frac{400}{s^2+12s+400}$
   b. $G(s) = \frac{900}{s^2+90s+900}$
   c. $G(s) = \frac{225}{s^2+30s+225}$
   d. $G(s) = \frac{625}{s^2+625}$
2. Given the transfer function

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

   Find $\boldsymbol{\zeta}$ and $\boldsymbol{\omega_n}$
3. Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

   Find T_p, $\%OS$, T_s
4. Consider the system with $\zeta = 0.8$, $\omega_n = 12$ rad/sec

   Calculate

   Delay time, rise time, overshoot, and settling time
5. $G(s) = \frac{100}{s^2+s+100}$

   Find ω_n, ζ, ω_d
6. The transfer function of a system is given as $361/(s^2 + 16s + 361)$. Find the following for a unit step input:

(a) Undamped natural frequency

(b) Damping ratio

(c) Damped natural frequency

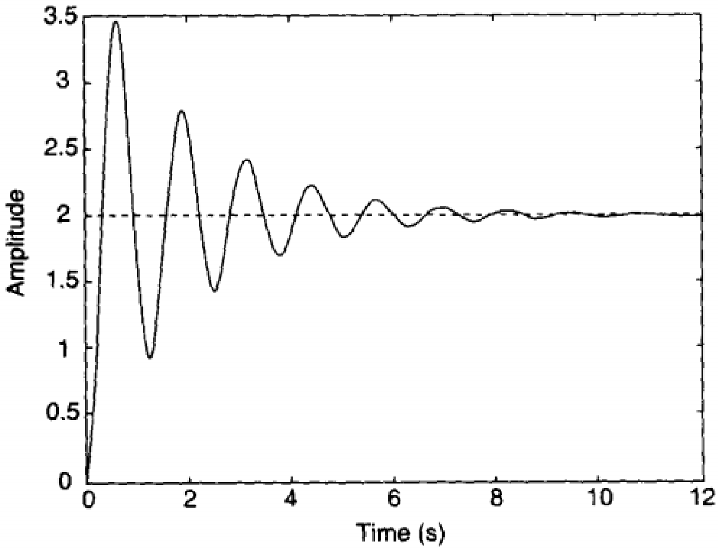
(d) Settling time

(e) Peak time

(f) Rise time

(g) Percentage overshoot

7. Figure below shows the unit-step response of a second-order system.



Determine the following from the plot:

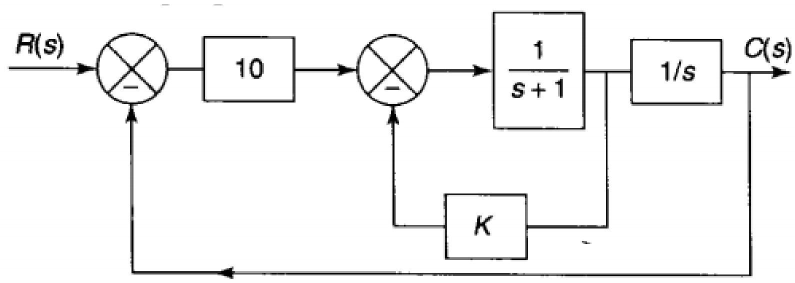
(a) The gain;

(b) Damping ratio,

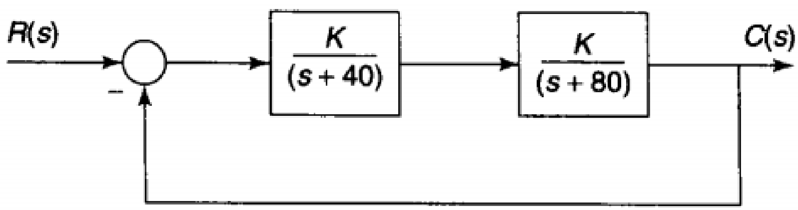
(c) Natural frequency,

(d) The transfer function

8. For the system shown below, determine the value of ***K*** such that the output signal will produce 10% overshoot in response to a step input



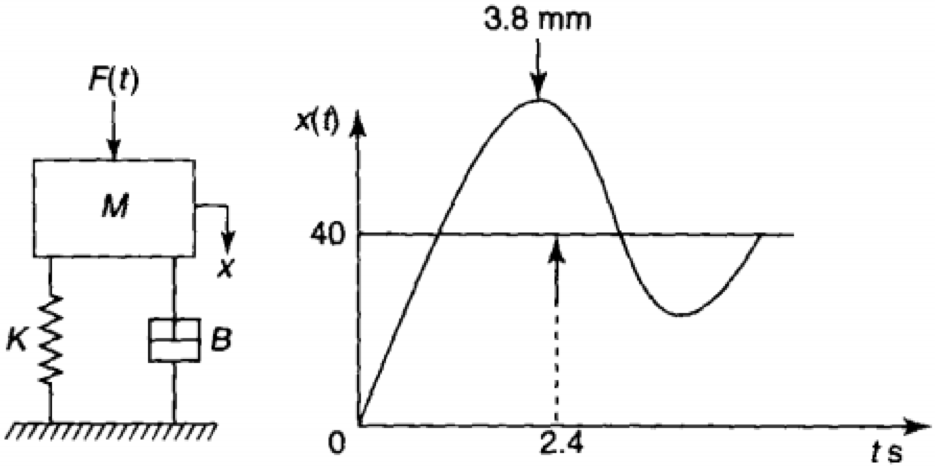
9. A servo mechanism is shown below



   (a) Calculate the value of ***K*** such that the natural frequency is 100 rad/sec

   (b) Determine the rise time

10. Figure below shows a mechanical vibratory system when $1\ kN$ force is applied to the system. The mass oscillates as shown. Determine the mass M, damping coefficient B and spring stiffness K of the system for this response curve


F(t)
M
x
K
B
3.8 mm
x(t)
40
0
2.4
t s