Chapter 07

Steady state error

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Introduction

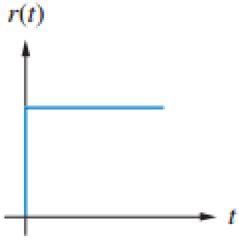
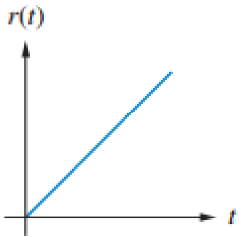
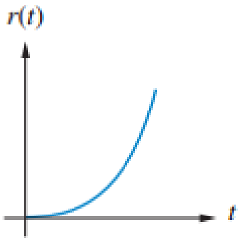
The static response of a system is the steady-state error that a system may exhibit between the controlled variable and the input

Whenever a system is distributed, or whenever an input is applied to a system, the system will be in **disturbed** state (**transient** state) for some time, and later on, it gets stabilized or attains a steady-state condition

The difference between the steady-state value and the reference value is known as the **steady state error** — it is a measure of the accuracy of a control system

> The **steady-state error** of a system response is defined as the discrepancy between the output and the reference input when the steady state ($t \to \infty$) is reached.

It should be pointed out that the **steady-state error may be defined for any test signal** such as a step-function, ramp-function, parabolic-function, or even a sinusoidal input

Waveform	Name	Physical interpretation
	Step	Constant position
	Ramp	Constant velocity
	Parabola	Constant acceleration

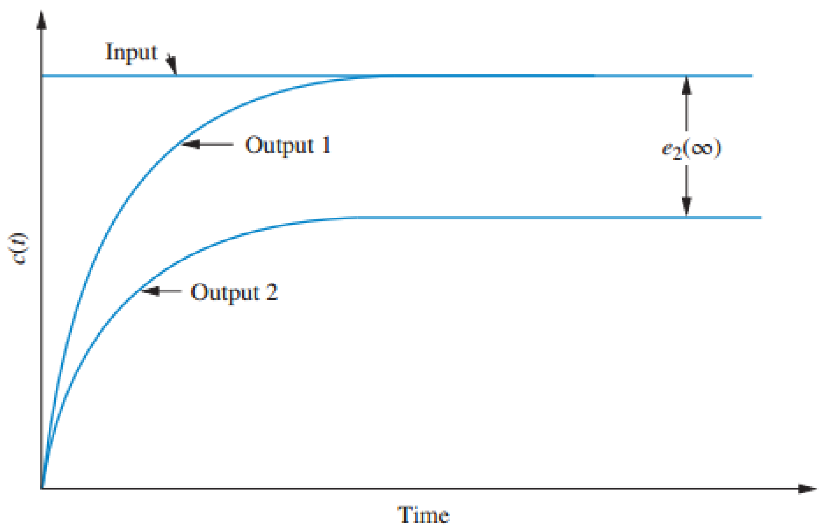
One of the objectives of most control systems is that the system output response follows a specific reference signal accurately in the steady state.

In the real world, **because of friction and other imperfections and the natural composition of the system**, the steady state of the output response seldom agrees exactly with the reference. Therefore, steady-state errors in control systems are almost unavoidable. **In a design problem**, one of the objectives is to keep the steady-state error to a minimum, or below a certain tolerable value, and at the same time the transient response must satisfy a certain set of specifications

Since we are concerned with the difference between the input and the output of a feedback control system after the steady state has been reached, **our discussion is limited to stable systems**, where the natural response approaches zero as Unstable systems represent loss of control in the steady state and are not acceptable for use at all.

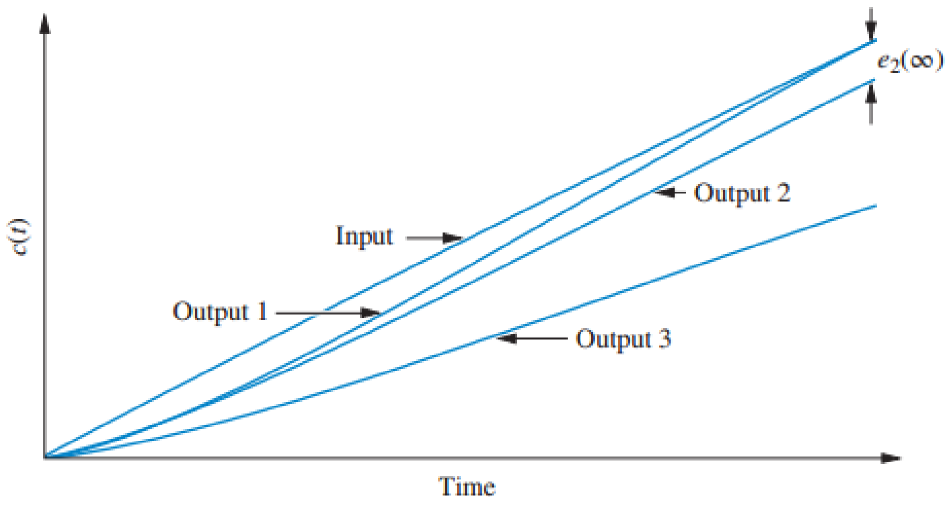
Thus, the engineer **must check the system for stability** while performing steady-state error analysis and design.

In Figure below a step input and two possible outputs are shown. Output 1 has zero steady-state error, and Output 2 has a finite steady-state error



Another example is shown below, where a ramp input is compared with Output 1, which has zero steady-state error, and Output 2, which has a finite steady-state error. Errors are measured vertically between the Input and Output 2 after the transients have died down.

For the ramp input another possibility exists. If the output's slope is different from that of the input, then Output 3, shown in Figure below, results. Here the steady-state error is infinite as measured vertically between the Input and Output 3 after the transients have died down, and t approaches infinity.



Many steady-state errors in control systems **arise from nonlinear sources**, such as backlash in gears or a motor that will not move unless the input voltage exceeds a threshold.

The steady- state errors we study here are errors that arise from the **configuration of the system itself** and the **type of applied input**.

Steady-state error (SSE) for open loop control system

$E(s) = R(s) - C(s)$

Also,

$$\frac{C(s)}{R(s)} = G(s)$$

$C(s) = R(s)G(s)$

$\therefore \quad E(s) = R(s) - R(s)G(s)$

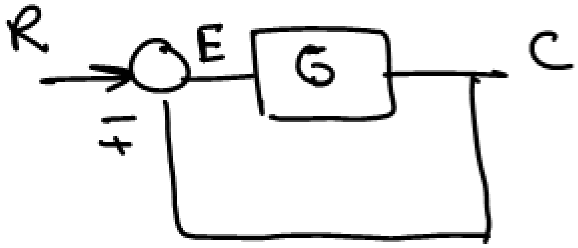
$E(s) = R(s)[1 - G(s)]$

$$e_{ss} = \lim_{s \to 0} s\,E(s)$$

$$\boldsymbol{e_{ss} = \lim_{s\to 0} s\, R(s)[1 - G(s)]}$$

Steady-state error (SSE) for closed loop control systems

Steady-state error can be calculated from a system's closed-loop transfer function, $T(s)$, or the open-loop transfer function, $G(s)$, for **unity feedback systems**



Here, $H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$C(s) = E(s)G(s)$$

$$\frac{E(s)G(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\boldsymbol{E(s) = \frac{R(s)}{1 + G(s)}}$$

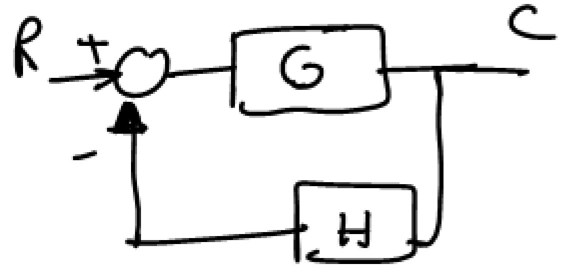
- Depends on input type, i.e. $\boldsymbol{R(s)}$
- Depends of system transfer function

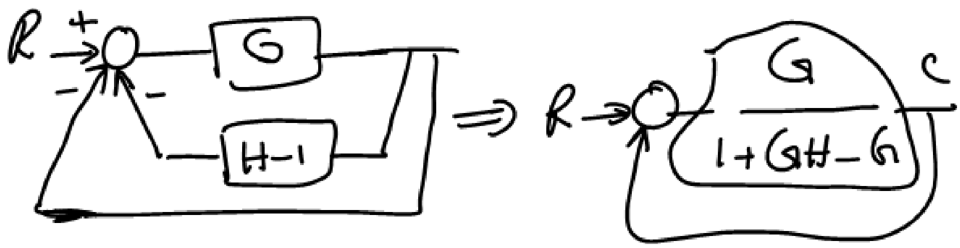
$\because \quad \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

$$\boldsymbol{e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)}}$$

✍ Note 01:

Any system can be described as unity feedback system to be able to obtain open-loop transfer function





Open loop transfer function

$$\boldsymbol{G_0(s) = \frac{G}{1+GH-G}}$$

✍ Note 02:

Transfer function

$$T = \frac{G_0}{1 + G_0}$$

$$T + TG_0 = G_0$$

$$T = G_0(1 - T)$$

$$G_0 = \frac{T}{1 - T}$$

So, given the transfer function for a system, you can get open loop transfer function G_0, then you can continue with other steps to get steady state error $\boldsymbol{e_{ss}}$

Find the steady-state error for the transfer function $$T(s) = \frac{5}{(s^2 + 7s + 10)}$$ and the input is a unit step

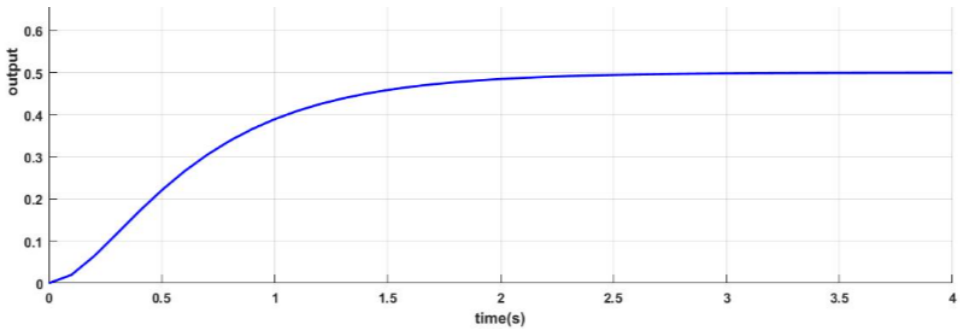
$$G_0 = \frac{5}{s^2 + 7s + 5}$$

$$e_{ss} = \lim_{s \to 0} s \frac{R(s)}{1 + G(s)}$$

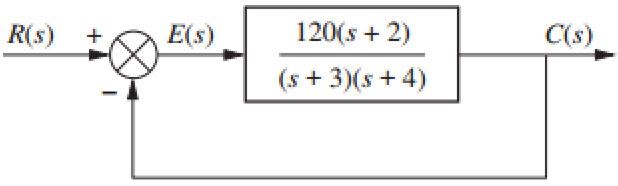
$$e_{ss} = \lim_{s\to 0} s\frac{1/s}{1+\dfrac{5}{s^2+7s+5}} = \lim_{s\to 0}\frac{s^2+7s+5}{s^2+7s+10}$$

$$e_{ss} = 0.5$$

```
a = [5];
b = [1 7 10];
sys = tf(a,b);
t = 0:0.1:4;
y = step(sys,t);
plot(t,y);
```



Find the steady state error for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown



$$G_0 = \frac{120(s+2)}{(s+3)(s+4)}$$

a. e_{ss} for input $5u(t)$

$$e_{ss} = \lim_{s\to 0} s \frac{5/s}{1 + \frac{120(s+2)}{(s+3)(s+4)}}$$

$$= \lim_{s\to 0} 5 * \frac{(s+3)(s+4)}{(s+3)(s+4) + 120(s+2)}$$

$$= 5 * \frac{12}{12 + 240} = 0.238$$

Also, for unit-step input, e_{ss} can be calculated from

$$e_{ss} = \lim_{s\to 0} s \frac{R(s)}{1 + G(s)} = \lim_{s\to 0} s \frac{1/s}{1 + G(s)}$$

$$e_{ss} = \frac{1}{1 + \lim_{s\to 0} G(s)}$$

For the numerical example above:

$$e_{ss} = \frac{5}{1 + \lim_{s\to 0} G(s)}$$

$$e_{ss} = \frac{5}{1 + \lim_{s\to 0} \frac{120(s+2)}{(s+3)(s+4)}} = \frac{5}{1 + \frac{120(2)}{(3)(4)}} = \frac{5}{21} = 0.238$$

b. e_{ss} for input $5tu(t)$

for unit-ramp input, e_{ss} can be calculated from

$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)} = \lim_{s\to 0} s\frac{1/s^2}{1+G(s)}$$

$$e_{ss} = \frac{1}{\lim_{s\to 0} sG(s)}$$

For the numerical example above:

$$e_{ss} = \frac{5}{\lim_{s\to 0} s\frac{120(s+2)}{(s+3)(s+4)}} = \infty$$

c. e_{ss} for input $5tu(t)$

for ramp input, e_{ss} can be calculated from

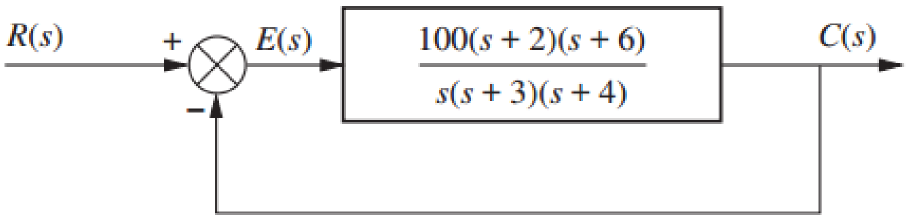
$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)} = \lim_{s\to 0} s\frac{2/s^3}{1+G(s)}$$

$$e_{ss} = \frac{2}{\lim_{s\to 0} s^2G(s)}$$

For the numerical example above:

$$e_{ss} = \frac{10}{\lim_{s\to 0} s^2\frac{120(s+2)}{(s+3)(s+4)}} = \infty$$

Find the steady state error for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown



(a) Steady-state error for $5\ u(t)$

$$e_{ss} = \frac{sR(s)}{1 + \lim_{s\to 0} G(s)}$$

$$e_{ss} = \frac{5}{1 + \lim_{s\to 0} G(s)}$$

$$e_{ss} = \frac{5}{1 + \lim_{s\to 0} \dfrac{100(s+2)(s+6)}{s(s+3)(s+4)}}$$

$$e_{ss} = 0$$

(b) Steady-state error for $5\ tu(t)$

$$e_{ss} = \frac{5}{\lim_{s\to 0} sG(s)}$$

$$e_{ss} = \frac{5}{\lim_{s\to 0} s\dfrac{100(s+2)(s+6)}{s(s+3)(s+4)}}$$

$$e_{ss} = \frac{5}{100 * \frac{2*6}{3*4}} = 0.05$$

(c) Steady-state error for 5 $t^2u(t)$

$$e_{ss} = \frac{10}{\lim_{s\to 0} s^2 G(s)}$$

$$e_{ss} = \frac{10}{\lim_{s\to 0} s^2 \frac{100(s+2)(s+6)}{s(s+3)(s+4)}} = \infty$$

A unity feedback system has the following forward transfer function

$$\frac{10(s+20)(s+30)}{s(s+25)(s+35)}$$

Find the steady-state error for the following inputs: $15\ u(t)$,$15t\ u(t)$, $15t^2\ u(t)$

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input

Control systems may be classified according to their **ability to follow step inputs, ramp inputs, parabolic inputs**, and so on. This is a reasonable classification scheme, because actual inputs may frequently be considered combinations of such inputs.

The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

If $G_0(s)$ can be written as

$$G_0(s) = \frac{k(s+z_1)(s+z_1)\dots(s+z_m)}{s^n(s+p_1)(s+p_2)\dots\left(s+p_q\right)}$$

$\boldsymbol{n}$: system type

$$G(s)H(s) = \frac{5(s+1)}{s^2(s+2)}$$

System type is 2

Note that this classification is different from that of the order of a system. **As the type number is increased**, **accuracy is improved**; however, **increasing the type number aggravates the stability problem**. A **compromise** between **steady-state accuracy** and **relative stability** is always necessary

- Static Error Constants

The static position error constant K_P “**use it when step input applied**” is defined by

$$K_P = \lim_{s \to 0} G(s) = G(0)$$

Thus, the steady-state error in terms of the static position error constant K_P is given by

$$e_{ss} = \frac{1}{1 + K_P}$$

The static velocity error constant K_v “**use it when ramp input applied**” is defined by

$$K_v = \lim_{s \to 0} sG(s)$$

Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

The static acceleration error constant K_a “**use it when parabolic input applied**” is defined by

$$K_a = \lim_{s \to 0} s^2 G(s)$$

Thus, the steady-state error in terms of the static velocity error constant K_a is given by

$$e_{ss} = \frac{1}{K_a}$$

The **static error constants** defined above are figures of merit of control systems. The higher the constants, the smaller the steady-state error.

In a given system, the output may be the position, velocity, pressure, temperature, or the like. The physical form of the output, however, is immaterial to the present analysis. Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on. This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on

The error constants K_P, K_v, and K_a describe the ability of a unity-feedback system to reduce or eliminate steady-state error. Therefore, they are indicative of the steady-state performance. It is generally desirable to increase the error constants, while maintaining the transient response within an acceptable range. It is noted that to improve the steady state performance we can increase the type of the system by adding an integrator or integrators to the feedforward path. This, however, introduces an additional stability problem.

Problems:

1. When the closed-loop system involves a numerator dynamic, the unit-step response curve may exhibit a large overshoot. Obtain the unit-step response of the following system with MATLAB:

$$\frac{C(s)}{R(s)} = \frac{10s + 4}{s^2 + 4s + 4}$$

   Obtain also the unit-ramp response with MATLAB

2. Consider a unity-feedback control system with the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{Ks + b}{s^2 + as + b}$$

   Determine the open-loop transfer function $G(s)$. Show that the steady-state error in the unit-ramp response is given by

$$e_{ss} = \frac{1}{K_v} = \frac{a - K}{b}$$

3. Consider a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(Js + B)}$$

   Discuss the effects that varying the values of K and B has on the steady-state error in unit-ramp response. Sketch typical unit-ramp response curves for a small value, medium value, and large value of K, assuming that B is constant.

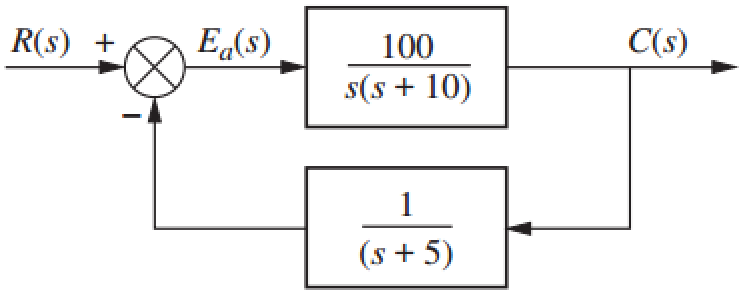
4. A unity feedback system has the following forward transfer function:

$$G(s) = \frac{K(s+12)}{(s+14)(s+18)}$$

Find the value of $\boldsymbol{K}$ to yield a 10% error in the steady state

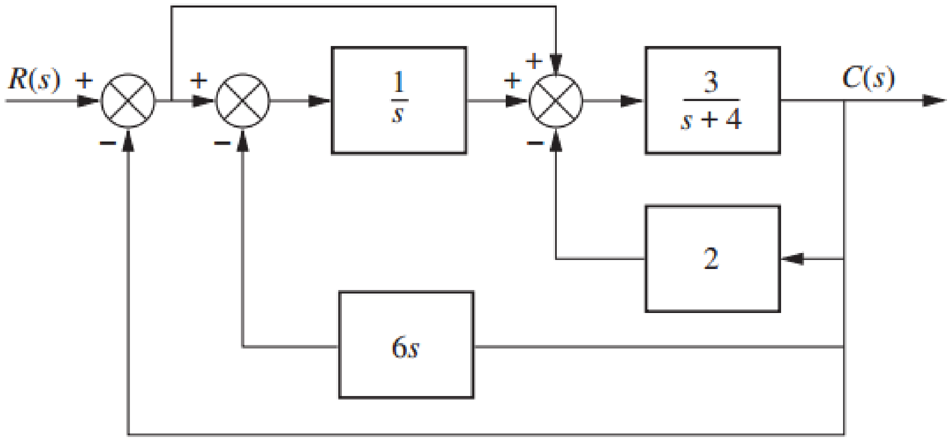
Ans. 189

5. For the system shown in Figure, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same

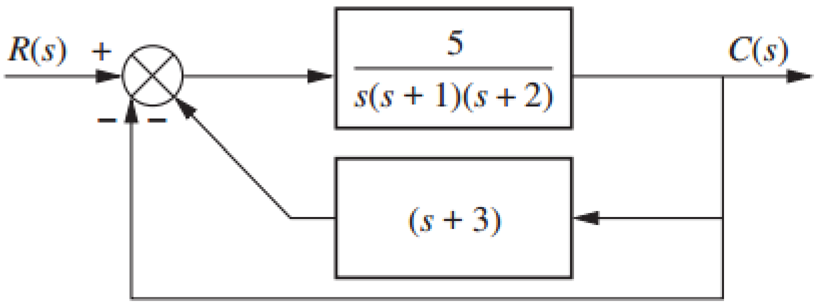


6. For the system shown in Figure, what steady-state error can be expected for the following test inputs:

$10u(t)$, $10tu(t)$, $10t^2u(t)$



7. For the system shown in Figure



(a) Find K_P, K_v, and K_a

(b) Find the steady-state error for an input of $50u(t)$, $50tu(t)$, $50t^2u(t)$

(c) State the system type.