Chapter 08

Stability analysis

Table of content

- **Introduction**

A **stable system** is defined as a system with a **bounded** (**limited**) system response. That is, if the system is subjected to a bounded input or disturbance and the **response is bounded in magnitude**, the system is said to be **stable**.

> A **stable system** is a dynamic system with a bounded response to a bounded input

A **necessary and sufficient condition** for a feedback system to be stable is that **all the poles of the system transfer function have negative real parts**

A system is not stable if not all the roots are in the left-hand plane

If the **characteristic equation** has **simple roots on the imaginary axis** ($j\omega$-axis) **with all other roots in the left half-plane**, the steady-state output will be **sustained oscillations**

For an **unstable system**, the characteristic equation has at least one root in the right half of the s-plane or repeated $j\omega$ roots; for this case, the output will become unbounded for any input

Consider that the characteristic equation of a linear time-variant **SISO** system is of the form

$$C/cs = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

where all the coefficients are real. **To ensure the last equation does not have roots with positive real parts, it is necessary (but not sufficient**) that the following conditions hold:

- All the coefficients of the equation have the same sign
- None of the coefficients vanish

However, these conditions are not sufficient, for it is quite possible that an equation with all its coefficients nonzero and of the same sign still **may not have all the roots in the left half of the s-plane**

- **ROUTH'S STABILITY CRITERION**

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system?

it was stated that a control system is stable if and only if all closed-loop poles lie in the left-half s plane. Most linear closed-loop systems have closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

where the a's and b's are constants and $\boldsymbol{m} \leq \boldsymbol{n}$. A simple criterion, known as **Routh's stability criterion**, enables us to determine the number of closed-loop poles that lie in the right-half

s plane without having to factor the denominator polynomial. (The polynomial may include parameters that MATLAB cannot handle)

consider

$$C/cs = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

A necessary and sufficient condition for stability if and only if all the elements in the first column of the Routh array have no sign change

$$\begin{array}{l|llll}
s^n & 1 & a_2 & a_4 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & \dots \\
s^{n-2} & b_1 & b_2 & b_3 & \dots \\
s^{n-3} & c_1 & c_2 & c_3 & \dots \\
\dots & \dots & \dots & \dots & \dots \\
s^2 & * & * & & \\
s & * & & & \\
s^0 & * & & &
\end{array}$$

$$b_1 = \frac{a_1 * a_2 - a_3 * 1}{a_1}$$

$$b_2 = \frac{a_1 * a_4 - a_5 * 1}{a_1}$$

$$b_3 = \frac{a_1 * a_6 - a_7 * 1}{a_1}$$

$$c_1 = \frac{b_1 * a_3 - b_2 * a_1}{b_1}$$

$$c_2 = \frac{b_1 * a_5 - b_3 * a_1}{b_1}$$

The number of roots in RHS is equal to the number of sign change in the 1st column

Example:

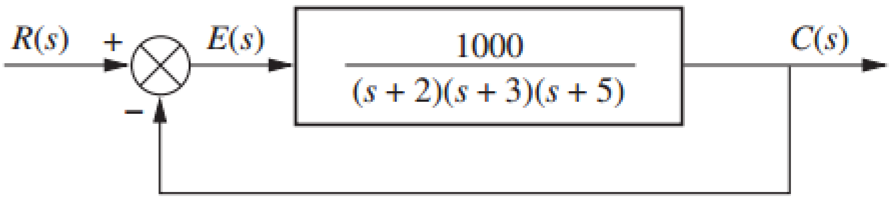
Given that the closed loop transfer function is

$$\frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\begin{array}{llll}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & \frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 & \frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 & \frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0 \\
s^1 & \frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & \frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0 & \frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0 \\
s^0 & \frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 & \frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0 & \frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0
\end{array}$$

Example:

Make the Routh table for the system shown



$$TF = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

s^3	1	31	0
s^2	10	1030	
s^1			
s^0			

For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below.

s^3	1	31	0
s^2	1	103	0
s^1	-72	0	0
s^0	103		

The number of sign change is 2, so there are two roots on the RHS.

The system is unstable

```
a = [1 10 31 1030];
roots(a)
```

-13.4136 + 0.0000i

1.7068 + 8.5950i

1.7068 - 8.5950i

Example:

C/cs. = $s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$

s^6	1	3	1	4
s^5	4	2	4	
s^4	2.5	0	4	
s^3	2	-2.4	0	
s^2	3	4		
s^1	-5.066			
s^0	4			

1	+
4	+
2.5	+
2	+
3	+

-5.066	-
4	+

The sign change is two, so we have two roots on the RHS

-3.2644 + 0.0000i

0.6797 + 0.7488i

0.6797 - 0.7488i

-0.6046 + 0.9935i

-0.6046 - 0.9935i

-0.8858 + 0.0000i

Example:

Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane

C/cs. = $3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$

s^7	3	6	7	2
s^6	9	4	8	6
s^5				
s^4				
s^3				
s^2				
s^1				

s^0	4

Example:

C/cs. = $s^4 + 2s^3 + 3s^2 + 4s + 5$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6		
s^0	5		

1	+	Two sign changes 2 roots on RHS 2 on the LHS
2	+	
1	+	
-6	-	
5	+	

Example:

C/cs. = $a_2 s^2 + a_1 s + a_0$

s^2	a_2	a_0
s^1	a_1	0

s^0	a_0	

Therefore, the requirement for a **stable second-order** system is simply that **all the coefficients be positive or all the coefficients be negative**

Example:

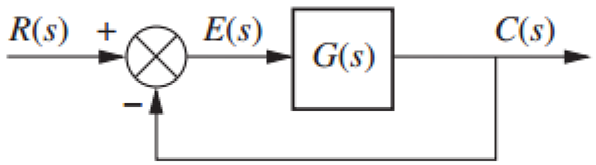
C/cs. = $a_3s^3 + a_2s^2 + a_1s + a_0$

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$\dfrac{a_2a_1 - a_0a_3}{a_2}$	
s^0	a_0	

For the third-order system to be stable, it is necessary and sufficient that the coefficients be positive and $a_2a_1 > a_0a_3$. The condition when $a_2a_1 = a_0a_3$ results in a marginal stability case, and one pair of roots lies on the imaginary axis in the s-plane

Problem:

For the unity feedback system of Figure



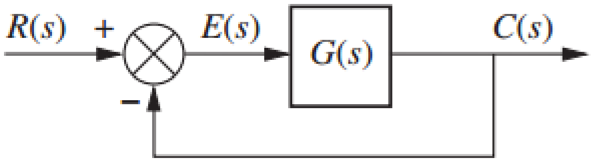
With

$$G(s) = \frac{K(s+6)}{s(s+1)(s+4)}$$

Determine the range of K to ensure stability

Problem:

For the unity feedback system of Figure



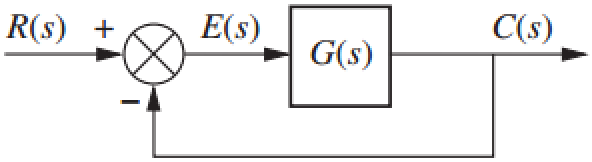
With

$$G(s) = \frac{K(s+3)(s+5)}{(s-2)(s-4)}$$

Determine the range of K to ensure stability

Problem:

Determine if unity feedback system of fig. below with



With

$$G(s) = \frac{K(s^2+1)}{(s+1)(s+2)}$$

Can be unstable

Special case:

If only the first element in one of the rows is zero, then we can replace the zero with a small positive constant $\epsilon > 0$ and proceed as follows:

(1) If the sign of the coefficient above the zero is the same as the one below it, it indicates that there are a pair of imaginary roots

(2) If the sign of the coefficient above the zero is opposite that below it, there is one change

Example:

C/cs. = $s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$

s^5	1	2	6
s^4	3	6	9
s^3	$\cancel{0}$ ϵ	3	
s^2	$6 - 9/\epsilon$	9	
s^1	$3 - \frac{9\epsilon^2}{6\epsilon - 9}$		

s^0	9

Take the limit as $\epsilon \to 0$

1	+		
3	+	1 sign change	Two sign changes
ϵ	ϵ		
$6-9/\epsilon$	-		
$3-\dfrac{9\epsilon^2}{6\epsilon-9}$	+	1 sign change	
9	+		

You may check by MATLAB

-2.9043 + 0.0000i

0.6567 + 1.2881i

0.6567 - 1.2881i

-0.7046 + 0.9929i

-0.7046 - 0.9929i

Example:

C/cs. = s^3+2s^2+s+2

s^3	1	1
s^2	2	2

$$\begin{array}{l|l} s^1 & \cancel{0}\ \epsilon \\ s^0 & 2 \end{array}$$

Pair of imaginary roots

-2.0000 + 0.0000i

0.0000 + 1.0000i

0.0000 - 1.0000i

Example:

C/cs. = $s^3 - 3s + 2$

$$\begin{array}{l|lll} s^3 & 1 & -3 & \text{Two sign} \\ s^2 & 0 & 2 & \text{changes} \\ s^1 & -3 - 2/\epsilon & & \text{2 roots on} \\ s^0 & 2 & & \text{RHS} \end{array}$$

✍ Note:

$$s^3 - 3s + 2 = (s-1)^2(s+2)$$

Example:

C/cs. = $s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$

s^5	1	11	28	Two
s^4	5	23	12	complex
s^3	6.4	25.6		poles on
s^2	**3**	12		imaginary
s^1	$\cancel{0}$ ϵ			axis
s^0	**12**			

Special case:

When an entire row of the Routh array is zero. This indicates that there are complex conjugate pairs of roots that are mirror images of each other with respect to the imaginary axis

Example:

C/cs. = $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50$

s^5	1	24	-25	
s^4	2	48	-50	Auxiliary equation
s^3	0	0	0	Row = 0
s^2	**3**	12		
s^1	$\cancel{0}$ ϵ			
s^0	**12**			

Auxiliary equation

P(s) = $2s^4 + 48s^2 - 50$

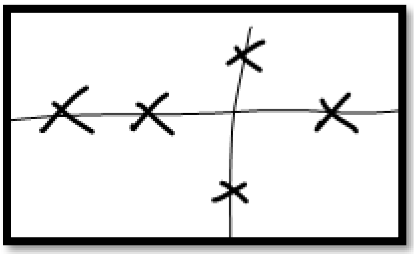
$$\frac{dP}{ds} = 8s^3 + 96s$$

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.67		
s^0	-50		

s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.67		
s^0	-50		

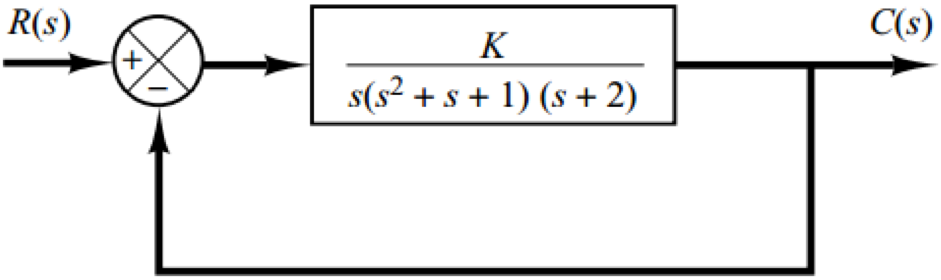
One sign change so we have one root on the RHS, so we have 1 on LHS→ 4 - 2 = 2 → imaginary

5 – 4 → 1 → LHS



Example:

Consider the system shown in Figure. Let us determine the range of K for stability. The closed-loop transfer function is



$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

s^4	1	3	K
s^3	3	2	0
s^2	7/3	K	
s^1	$2 - \frac{9}{7}K$		
s^0	K		

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$