Probability

Counting Methods

Prof. Dr. Mostafa Elhosseini

https://youtube.com/drmelhosseini

Counting

- When **computing probabilities**, it is sometimes **necessary** to **determine the number of outcomes** in a sample space
- In this lecture we will describe **several methods for doing this**.
- The **basic rule**, which we will call the **fundamental principle of counting**

Example

- A certain make of automobile is available in any of **three colors**: red, blue, or green, and comes with either a **large or small engine**. In **how many ways** can a buyer choose a car?

Example

- There are **three choices of color** and **two choices of engine**. A complete list of choices is written in the following **3 × 2** table. The total number of choices is **(3)(2) = 6**.

	Red	Blue	Green
Large	Red, Large	Blue, Large	Green, Large
Small	Red, Small	Blue, Small	Green, Small

Example

- if there are $\boldsymbol{n_1}$choices of color and $\boldsymbol{n_2}$ choices of engine, a complete list of choices can be written in an $\boldsymbol{n_1} \times \boldsymbol{n_2}$ table, so the total number of choices is $\boldsymbol{n_1 n_2}$

Fundamental principle of counting

- Assume that k operations are to be performed. If there are n_1 ways to perform the first operation, and if for each of these ways there are n_2 ways to perform the second operation, and if for each choice of ways to perform the first two operations there are n_3 ways to perform the third operation, and so on, then the total number of ways to perform the sequence of k operations is $n_1 n_2 \cdots n_k$.

Example 01

- When ordering a certain type of computer, there are **3** choices of hard drive, **4** choices for the amount of memory, **2** choices of video card, and **3** choices of monitor. <u>In how many ways can a computer be ordered?</u>

Example

- When ordering a certain type of computer, there are 3 choices of hard drive, 4 choices for the amount of memory, 2 choices of video card, and 3 choices of monitor. <u>In how many ways can a computer be ordered?</u>

- The total number of ways to order a computer is (3)(4)(2)(3) = 72.

Permutations

- A **permutation** “ التبديل “is an ordering of a collection of objects. For example, there are **six permutations** of the letters **A, B, C**:
  - ABC, ACB, BAC, BCA, CAB, and CBA.
- With only three objects, it is easy to determine the number of permutations just by listing them all

Permutations – How to count?

- The **fundamental principle of counting** can be used to determine the number of permutations of any set of objects.
- For example, we can determine the number of permutations of a set of three objects as follows. **There are 3 choices for the object to place first**. After that choice is made, there are **2 choices remaining for the object to place second**. Then there is **1 choice left** for the object to place last.
- Therefore, the total number of ways to order three objects is **(3)(2)(1) = 6**.

Permutations – How to count?

- The number of permutations of n objects is **n!**

Example

- Five people stand in line at a movie theater. **Into how many different orders can they be arranged?**

Example

- Five people stand in line at a movie theater. **Into how many different orders can they be arranged?**

- The number of permutations of a collection of five people is

5! = (5)(4)(3)(2)(1) = 120.

Note…

- Sometimes we are interested in counting the **number of permutations of subsets of a certain size** chosen from a larger set

Example

- Five **lifeguards** “ منقذين“ are available for duty one Saturday afternoon. There are three lifeguard stations. In how many ways can three lifeguards be chosen and ordered among the stations?

Example

- Five lifeguards are available for duty one Saturday afternoon. There are three lifeguard stations. In how many ways can three lifeguards be chosen and ordered among the stations?

- We use the fundamental principle of counting. There are 5 ways to choose a lifeguard to occupy the first station, then 4 ways to choose a lifeguard to occupy the second station, and finally 3 ways to choose a lifeguard to occupy the third station. The total number of permutations of three lifeguards chosen from 5 is therefore **(5)(4)(3) = 60**

Permutations

- The number of permutations of k objects chosen from a group of n objects is

$$\frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})!}$$

Combinations

- In some cases, when choosing a set of objects from a larger set, we **don’t care about the ordering** of the chosen objects; we **care only which objects are chosen**.
- For example, we **may not care which lifeguard occupies which station**; we might **care only which three lifeguards are chosen**.
- **Each distinct group of objects that can be selected, without regard to order, is called a combination**

Combinations

- In that example, we showed that there are **60 permutations of three objects chosen from five**. Denoting the objects, A, B, C, D, E, the figure presents a list of all 60 permutations.

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

Combinations

- The 60 permutations in Figure are arranged in 10 columns of 6 permutations each. Within each column, the three objects are the same, and the column contains the six different permutations of those three objects. Therefore, **each column represents a distinct combination** of three objects chosen from five, and there are 10 such combinations

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

Combinations

- The figure thus shows that the number of combinations of three objects chosen from five can be found by **dividing the number of permutations** of three objects chosen from five, which is **$5!/(5-3)!$**, by the number of **permutations of three objects**, which is **$3!$**
- In summary, the number of combinations of three objects chosen from five is **$5!/3!\,(5-3)!$**,

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

Combinations

- The number of combinations of k objects chosen from a group of n objects is
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Example

- At a certain event, 30 people attend, and 5 will be chosen at random to receive door prizes. The prizes are all the same, so the order in which the people are chosen does not matter. How many different groups of five people can be chosen?

Example

- At a certain event, 30 people attend, and 5 will be chosen at random to receive door prizes. The prizes are all the same, so the order in which the people are chosen does not matter. How many different groups of five people can be chosen?
- Since the order of the five chosen people does not matter, we need to compute the number of combinations of 5 chosen from 30. This is
- $\binom{\mathbf{30}}{\mathbf{5}} = \frac{\mathbf{30!}}{\mathbf{5!25!}} = \frac{(\mathbf{30})(\mathbf{29})(\mathbf{28})(\mathbf{27})(\mathbf{26})}{(\mathbf{5})(\mathbf{4})(\mathbf{3})(\mathbf{2})(\mathbf{1})} = \mathbf{142,506}$
- Choosing a combination of k objects from a set of n divides the n objects into two subsets: the k that were chosen and the $n - k$ that were not chosen

Example

- Assume that in a class of 12 students, a project is assigned in which the students will work in groups. Three groups are to be formed, consisting of five, four, and three students. We can calculate the number of ways in which the groups can be formed as follows

Example

- We consider the process of dividing the class into three groups as a sequence of two operations. The first operation is to select a combination of 5 students to make up the group of 5. The second operation is to select a combination of 4 students from the remaining 7 to make up the group of 4. The group of 3 will then automatically consist of the 3 students who are left
- The number of ways to perform the first operation is $\binom{12}{5} = \frac{12!}{5!7!}$
- After the first operation has been performed, the number of ways to perform the second operation is $\binom{7}{4} = \frac{7!}{4!3!}$
- The total number of ways to perform the sequence of two operations is therefore $\frac{12!}{5!7!}\frac{7!}{4!3!}$ = 27,720

Example

- The number of ways of dividing a group of n objects into groups of $k_1, \ldots, k_r$ objects, where $k_1 + \cdots + k_r = n$, is

$$\frac{n!}{k_1! \ldots k_r!}$$

Example

- A die is rolled 20 times. Given that three of the rolls came up 1, five came up 2, four came up 3, two came up 4, three came up 5, and three came up 6, how many different arrangements of the outcomes are there?

Example

- A die is rolled 20 times. Given that three of the rolls came up 1, five came up 2, four came up 3, two came up 4, three came up 5, and three came up 6, how many different arrangements of the outcomes are there?
- There are 20 outcomes. They are divided into six groups, namely, the group of three outcomes that came up 1, the group of five outcomes that came up 2, and so on. The number of ways to divide the 20 outcomes into six groups of the specified sizes is

$$\frac{20!}{3!\,5!\,4!\,2!\,3!\,3!} = 1.955 \times 10^{12}$$

References

- Statistics for Engineers and Scientists, Fourth Edition, William Navidi
  Colorado School of Mines
- Khan Academy