

Propositional Equivalences

Replacing a statement with another that says exactly the same thing

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What this section covers

01

Tautology, contradiction, contingency

Classifying compound propositions by the truth values they can take.

02

Logical equivalence

$p \equiv q$, and how a truth table settles it.

Tautologies and contradictions

Some compound propositions never change their mind.

Three kinds of compound proposition

Tautology تحصيل حاصل

Always **true**, no matter what truth values the propositional variables take.

Contradiction تضاد

Always **false**, whatever the variables do.

Contingency احتمال

Neither of the above — its value depends on the input.

- The traffic light is red, or the traffic light is not red.
- $(R \rightarrow S) \vee R$
- $(R \rightarrow S) \rightarrow (\neg S \rightarrow \neg R)$
- The traffic light is red AND the traffic light is not red at the same time.

One variable is enough

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

$$p \vee \neg p$$

True in both rows — a **tautology**. Either it is raining or it is not.

$$p \wedge \neg p$$

False in both rows — a **contradiction**. It cannot be raining and not raining at once.

Logical equivalence

Two expressions, one meaning — and the licence to swap them inside any argument.

Logically equivalent propositions

The compound propositions p and q are **logically equivalent** if $p \leftrightarrow q$ is a tautology. We write $p \equiv q$.

Equivalently: they have the **same truth value in every possible case** . The symbol $\leftrightarrow$ is sometimes used in place of $\equiv$.

$\equiv$ is not a connective

$$p \leftrightarrow q$$

A **compound proposition** . It has its own truth value, row by row.

$$p \equiv q$$

A **claim about** those propositions: that $p \leftrightarrow q$ is a tautology.
Not itself a proposition.

De Morgan's laws

TABLE 2 De Morgan's Laws.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Proving a De Morgan law by table

$\neg (p \vee q) \equiv \neg p \wedge \neg q$

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.						
p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Every conditional is a disjunction

$$p \rightarrow q \equiv \neg p \vee q$$

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Each new variable doubles the table

2

rows for 1 variable

4

rows for 2 variables

8

rows for 3 variables

2^n

rows for n variables

The distributive law on eight rows

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Augustus De Morgan



A noted teacher who **stressed principles over techniques** , and an extremely prolific writer — more than 1000 articles in over 15 periodicals.

In the 1840s he made fundamental contributions to symbolic logic, inventing notations that helped him prove propositional equivalences — including the laws now named after him.

In 1838 he gave perhaps the first clear explanation of **mathematical induction** — a term he coined. Among his students was Augusta Ada, Countess of Lovelace.

The catalogue of equivalences

Tables 6, 7 and 8 — the laws you will quote rather than re-derive.

The eleven named laws

TABLE 6 Logical Equivalences.

<i>Equivalence</i>	<i>Name</i>
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Identity

Adding a certainty changes nothing.

Idempotent

Saying it twice adds nothing.

Commutative

Order of the two sides is free.

Distributive

Multiply out, either direction.

Absorption

The weaker clause disappears.

Domination

A certainty swallows the rest.

Double negation

Two flips return you home.

Associative

Grouping of three is free.

De Morgan

Negate, then swap the connective.

Negation

p with $\neg p$ gives certainty either way.

Equivalences with conditionals

TABLE 7 Logical Equivalences
Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Equivalences with biconditionals

TABLE 8 Logical
Equivalences Involving
Biconditional Statements.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

De Morgan's laws for many propositions

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

and

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n).$$

Negating English sentences

$p \wedge q$

“Miguel has a cellphone **and** he has a laptop computer.”

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

“Miguel does not have a cellphone **or** he does not have a laptop computer.”

$r \vee s$

“Heather will go to the concert **or** Steve will go to the concert.”

$\neg(r \vee s) \equiv \neg r \wedge \neg s$

“Heather will not go to the concert **and** Steve will not go to the concert.”

ANALOGY To defeat a claim that you own **both** items, it is enough to be missing **one**. To defeat a claim that **at least one** person will come, you must account for **everybody**.