Automatic control systems

Dr. Mostafa Abdelkhalik Elhosseini

Associate Professor

Computers Engineering & Control Systems Dept.

Mansoura University

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Chapter 01

Introduction

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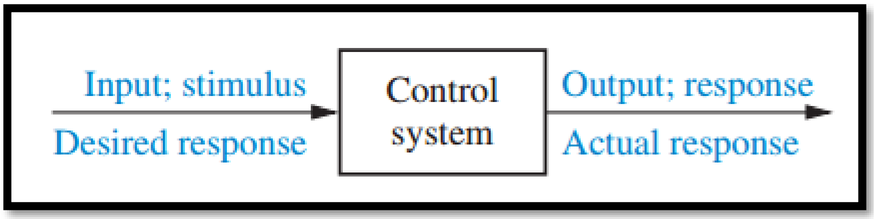
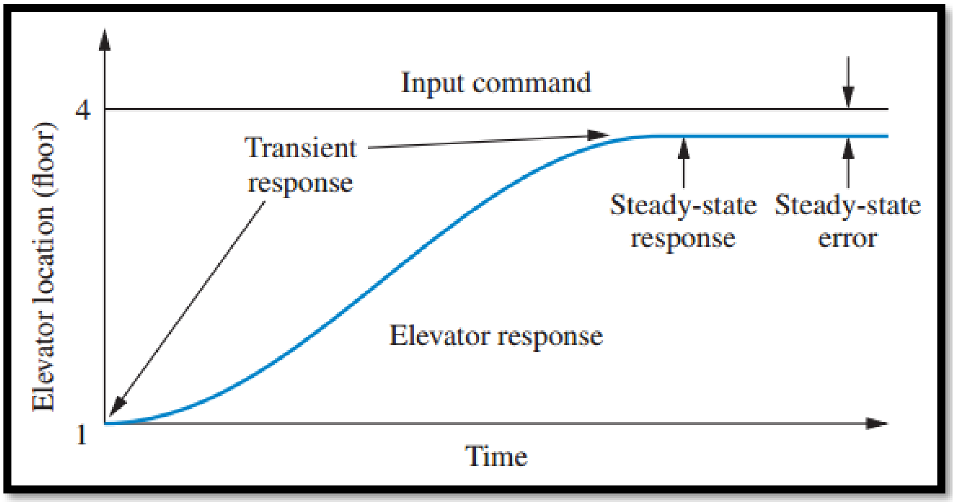
■ Control systems

- The basic strategy by which a control system operates is logical and natural. In fact, the same strategy is employed in **living organisms** to **maintain temperature**, **fluid flow rate**, and a host of other **biological functions**. This is natural process control.
- The technology of artificial control was first developed using a **human** as an integral part of the control action. When we learned how to use machines, electronics, and computers to replace the human function, the term **automatic control** came into use
- A control system consists of subsystems and processes (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input. Figure below shows a control system in its simplest form, where the input represents a desired output



- For example, consider an elevator. When the fourth-floor button is pressed on the first floor, the elevator rises to the fourth floor with a speed and floor leveling accuracy designed for passenger comfort. The push of the fourth-floor button is an input that represents our desired output, shown as a step function in Figure below.

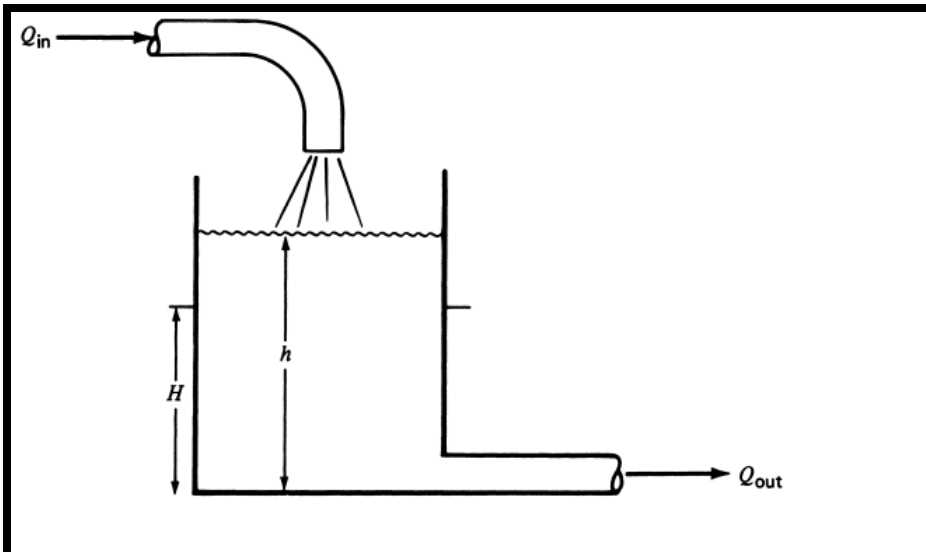


- The performance of the elevator can be seen from the elevator response curve in the figure.
- Two major measures of performance are apparent: (1) the **transient response** and (2) the **steady-state error**. In our example, **passenger comfort and passenger patience** are **dependent upon the transient response**. If this response is too fast, passenger comfort is sacrificed; if too slow, passenger patience is sacrificed. The steady-state error is another important performance specification since passenger safety and convenience would be sacrificed if the elevator did not level properly.

Process-Control Principles

In process control, the **basic objective is to regulate the value of some quantity**. To **regulate** means to **maintain that quantity at some desired value regardless of external influences**. The **desired value** is called the **reference value or setpoint**.

Figure below shows the process to be used for this discussion. Liquid is flowing into a tank at some rate, Q_{in} , and out of the tank at some rate, Q_{out}. The liquid in the tank has some height or level, h. It is known that the output flow rate varies as the square root of the height, $Q_{out} = K\sqrt{h}$, so the higher the level, the faster the liquid flows out. If the output flow rate is not exactly equal to the input flow rate, the level will drop, if $Q_{out} > Q_{in}$,or rise, if $Q_{out} < Q_{in}$.

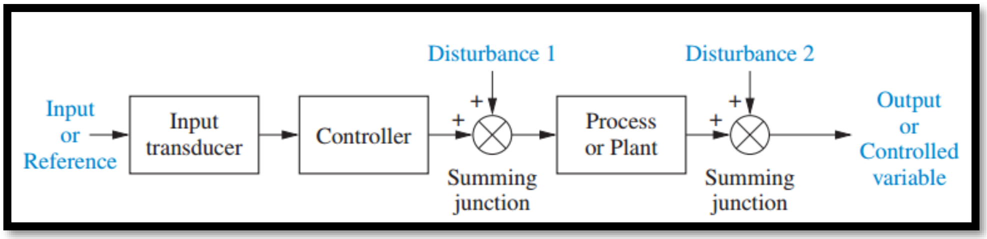


The objective is to regulate the level of liquid in the tank, *h*, to the value *H*.

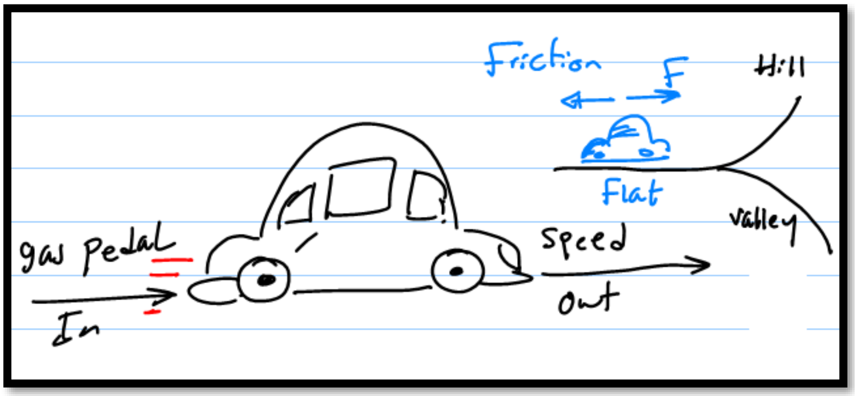
This process has a property called **self-regulation**. This means that for some input flow rate, the liquid height will rise until it reaches a height for which the output flow rate matches the input flow rate. A **self-regulating system does not provide regulation of a variable to any particular reference value**. In this example, the liquid level will adopt some value for which input and output flow rates are the same, and there it will stay. But if the input flow rate changed, then the level would change also, so it is not regulated to a reference value.

Open-loop systems

- Open loop control uses a controller and an actuator to obtain the desired response
- System without feedback
- It starts with a subsystem called an **input transducer**, which converts the form of the input to that used by the **controller**. The controller drives a process or a plant. The input is sometimes called the **reference**, while the **output** can be called the **controlled variable**

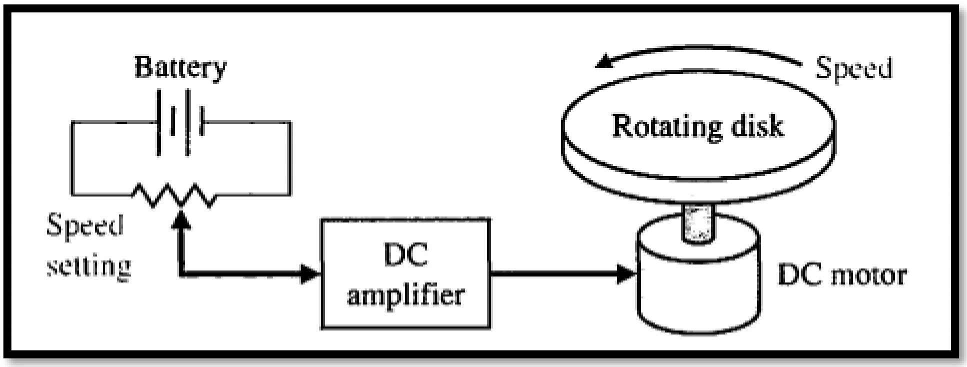


- The distinguishing characteristic of an open-loop system is that it cannot compensate for any disturbances that add to the controller's driving signal
- For example, **toasters** are open-loop systems, as anyone with burnt toast can attest. The controlled variable (output) of a toaster is the color of the toast. The device is designed with the assumption that the toast will be darker the longer it is subjected to heat. The toaster does not measure the color of the toast; it does not correct for the fact that the toast is rye, white, or sourdough, nor does it correct for the fact that toast comes in different thicknesses
- **Car driver control systems**: By pressing the **gas pedal** (**control input**) she hopes that the car's speed will come up to the desired value, but has no means of verifying the actual increase in speed. Such a control system, in which the **control input is applied without the knowledge of the plant output**, is called an **open-loop control system**

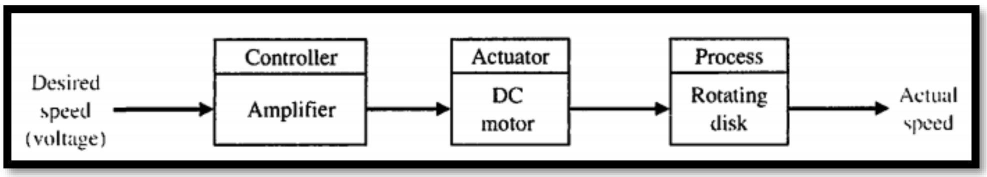


Imagine trying to obtain a constant speed of your car without benefit from built in automatic cruise control What happen when a car encounters a hill, or a valley? Without varying the input "gas pedal" the car will slow down or speed up and the desired constant speed will not be maintained ➔ problem of open loop — drawback

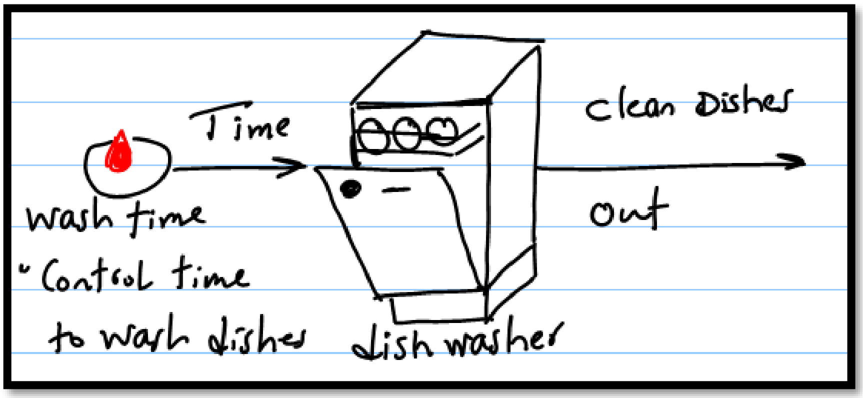
- **Rotating disk speed control**: This system uses a battery source to provide a voltage that is proportional to the desired speed. This voltage is amplified and applied to the motor.



- The block diagram of the open-loop system identifying the controller, actuator, and process is shown in figure below

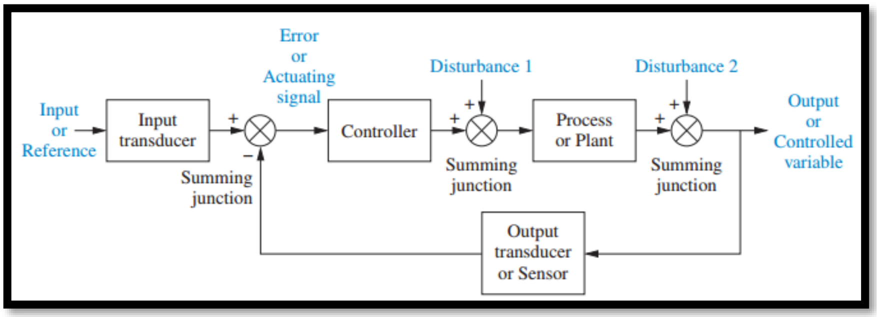


- **Dish washer:**



The goal of dish washer is to clean the dishes, which is the output, once the user sets the wash time, which control the time to clean dishes, the dish washer will run to that set time. Now this is happened regardless of the cleanness of the dishes

The disadvantages of open-loop systems, namely sensitivity to disturbances and inability to correct for these disturbances, may be overcome in **closed-loop systems**. The generic architecture of a closed-loop system is shown in figure below

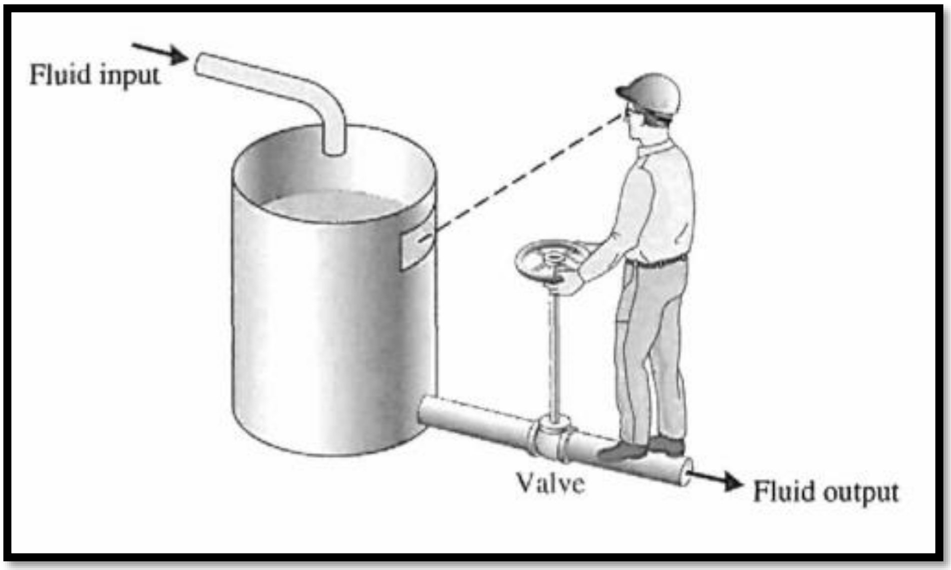


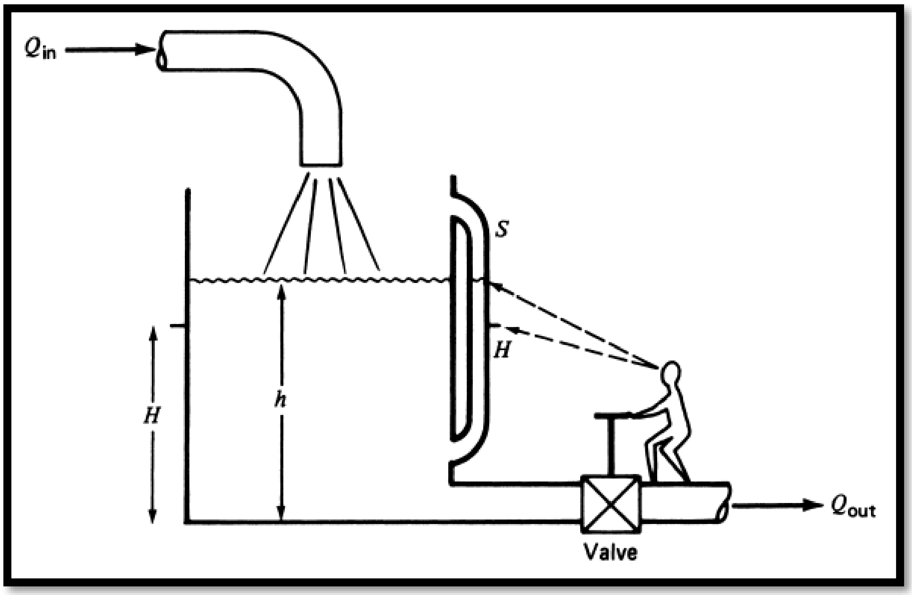
The input transducer converts the form of the input to the form used by the controller. An output transducer, or sensor, measures the output response and converts it into the form used by the controller. For example, if the controller uses electrical signals to operate the valves of a temperature control system, the input position and the output temperature are converted to electrical signals. The input position can be converted to a voltage by a potentiometer, a variable resistor, and the output temperature can be converted to a voltage by a thermistor, a device whose electrical resistance changes with temperature.

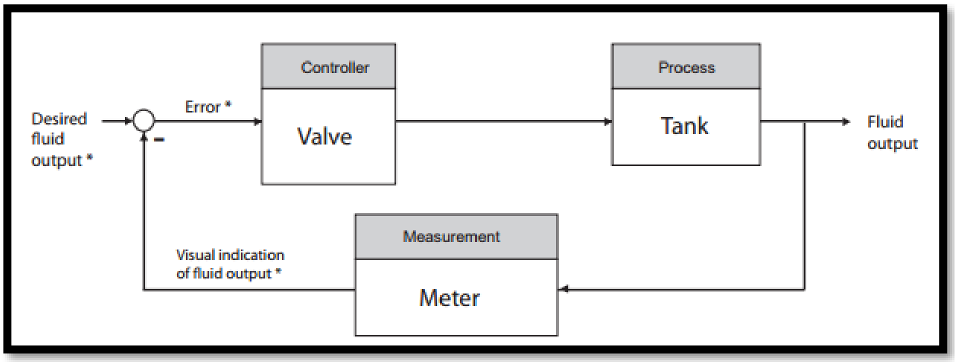
Closed-loop systems

- The **closed-loop system** compensates for disturbances by measuring the output response, feeding that measurement back through a **feedback** path, and **comparing** that response to the input at the summing junction. If there is any **difference** between the two responses, the system drives the plant, via the **actuating signal**, to make a correction. If there is no difference, the system does not drive the plant, since the plant's response is already the desired response

- **Closed-loop systems**, then, have the obvious advantage of greater accuracy than open-loop systems. They are **less sensitive to noise**, **disturbances**, and changes in the environment.
- **Transient response** and **steady-state error** can be controlled more conveniently and with greater flexibility in closed-loop systems, often by a simple adjustment of gain (**amplification**) in the loop and sometimes by **redesigning the controller**
- On the other hand, **closed-loop systems are more complex** and expensive than open-loop systems. A standard, open-loop **toaster** serves as an example: It is simple and inexpensive.
- A **closed-loop toaster** oven is more complex and more expensive since it has to **measure both color** (through light reflectivity) and **humidity** inside the toaster oven. Thus, the control systems engineer must consider the **trade-off** between the simplicity and low cost of an open-loop system and the accuracy and higher cost of a closed-loop system.
- In the past, control systems used a human operator as part of closed -loop control system

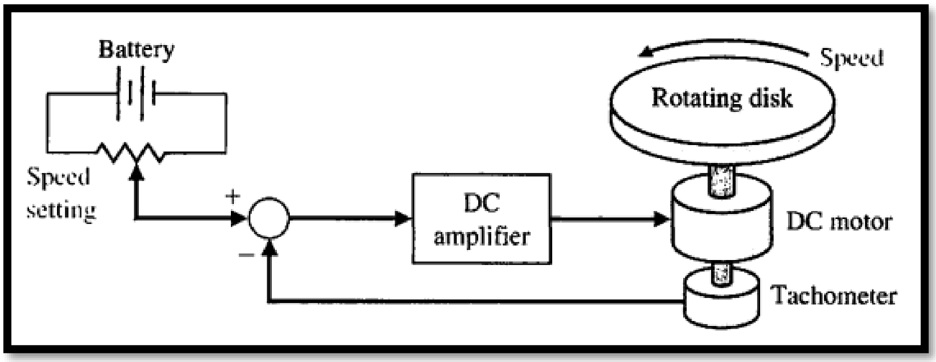

Fluid input
Valve
Fluid output


Q_{in}
S
H
H
h
Valve
Q_{out}

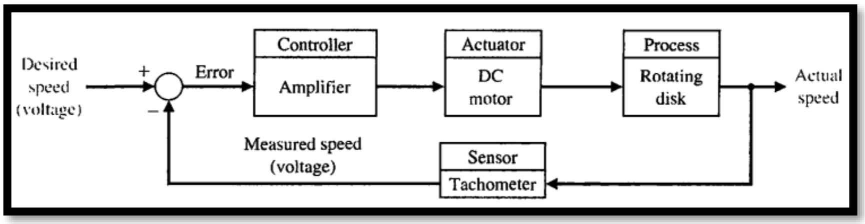


Rotating disk speed control

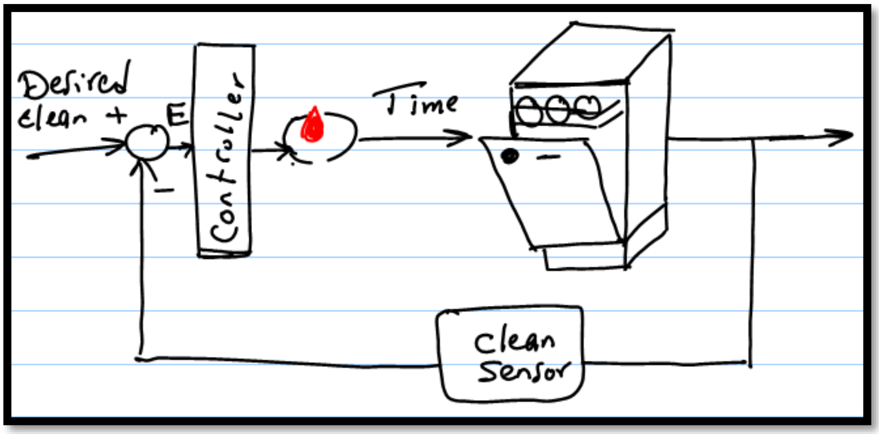
- To obtain a feedback system, we need to select a sensor. One useful sensor is a **tachometer** that provides an output voltage proportional to the speed of its shaft. Thus the closed-loop feedback system takes the form shown in Fig. below.



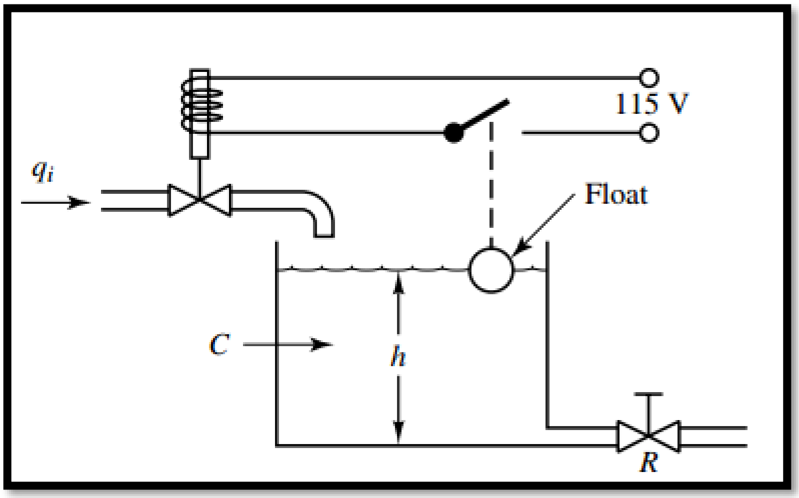
- The block diagram model of the feedback system is shown in Fig. below. The error voltage is generated by the difference between the input voltage and the tachometer voltage.



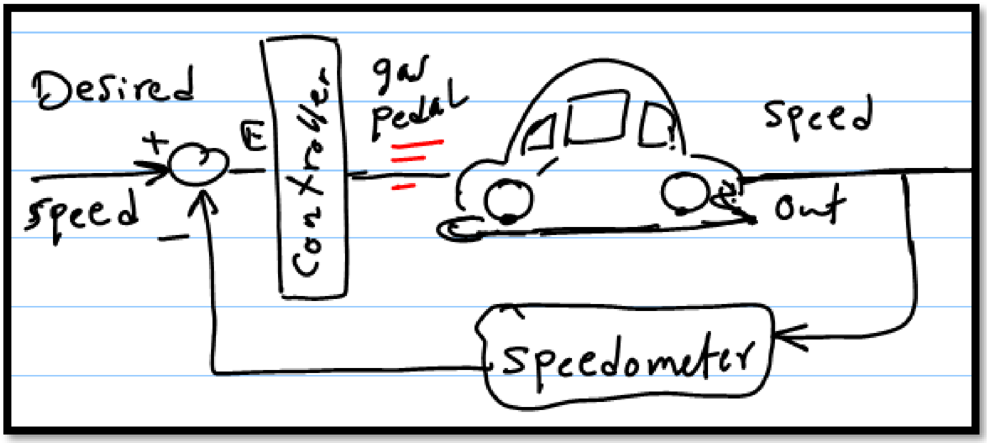
Dish washer:



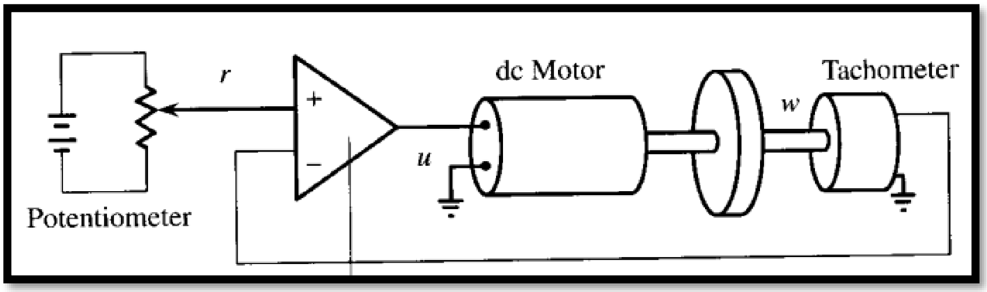
Liquid level control:

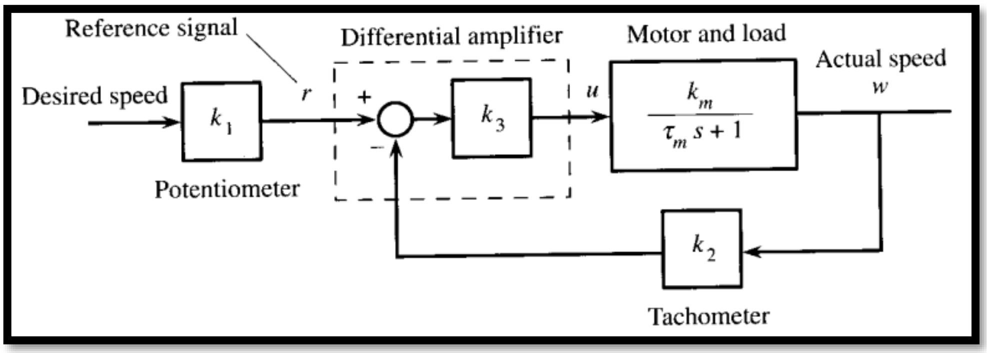


Car driver control systems:



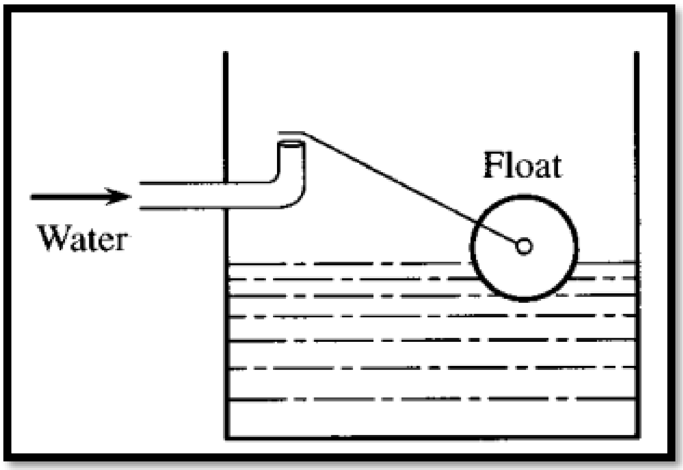
Velocity control system:

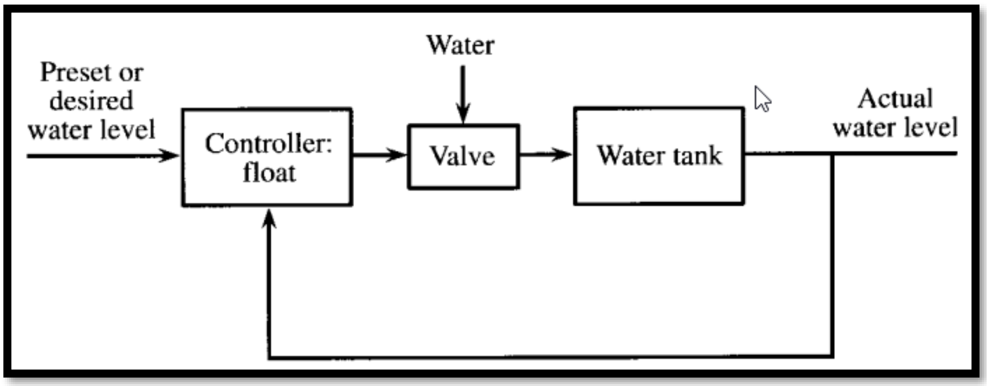


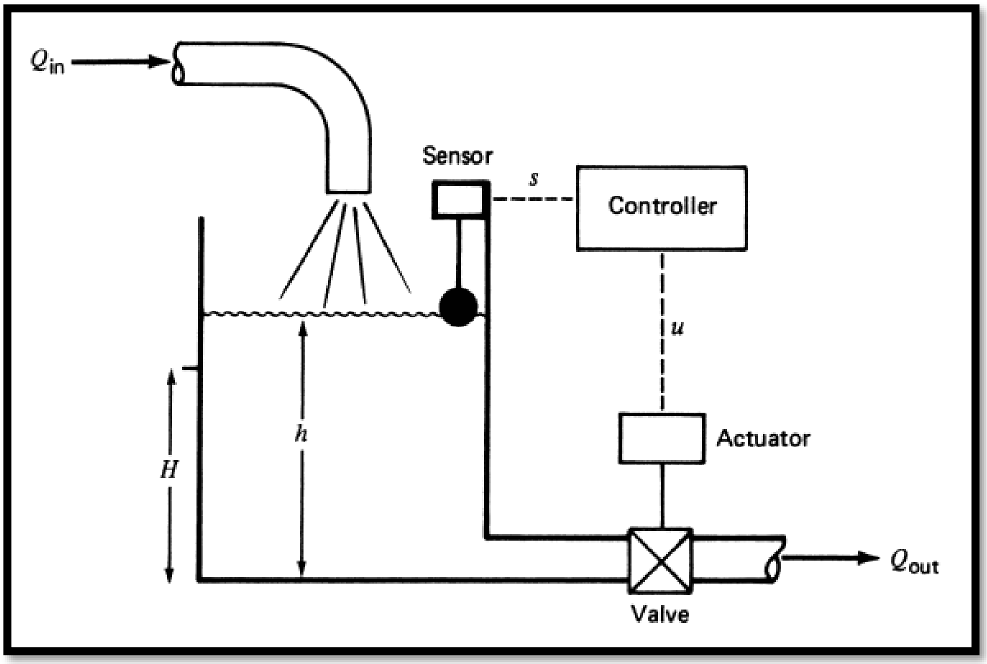


Bathroom toilet tank

The mechanism is designed to close the valve automatically whenever the water level reaches a preset height

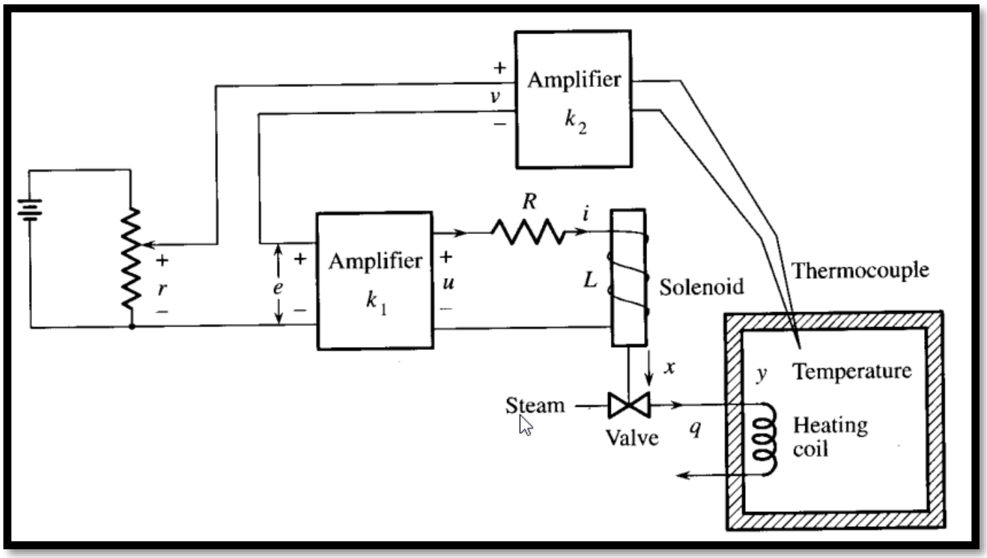






■ Temperature control

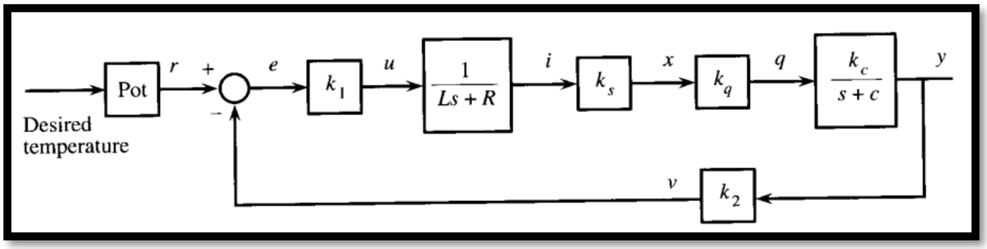
Temperature control is essential in many chemical processes. Consider the temperature control system shown below. The problem is to control the temperature inside the chamber. The chamber is heated by steam. The flow q of hot steam is proportional to the valve opening x; that is $q = k_q x$. The valve opening x is controlled by a solenoid and assumed to be proportional to the solenoid current i; that is $x = k_s i$.



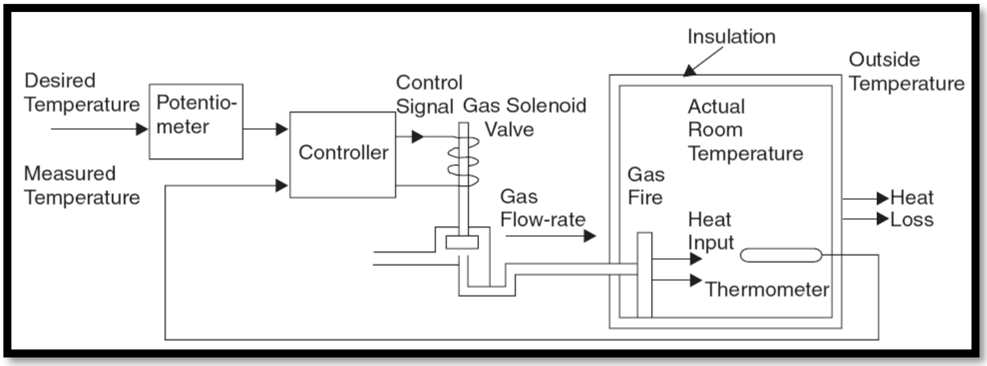
It is assumed that the chamber temperature y and the steam are related by

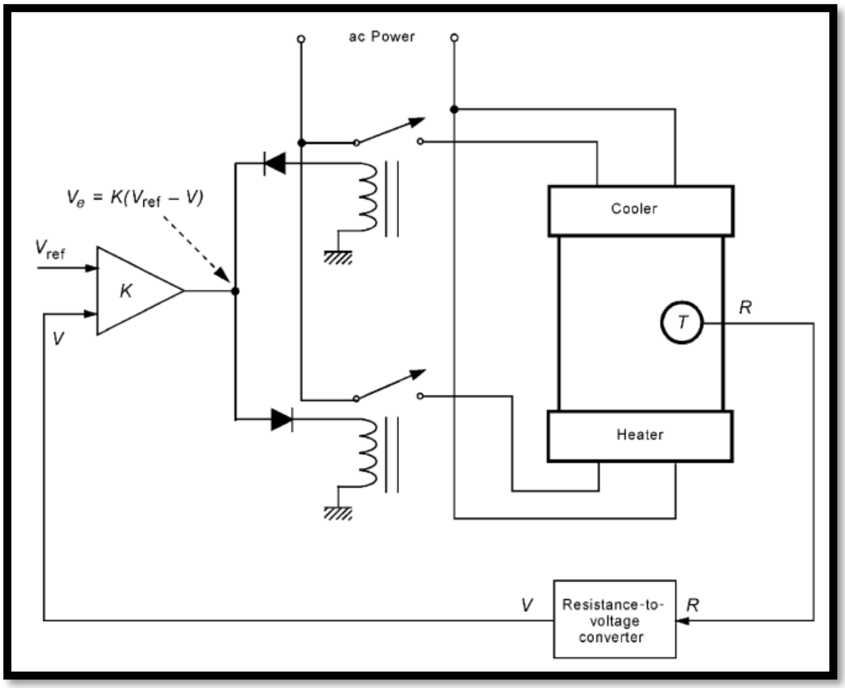
$$\frac{dy}{dt} = -cy + k_c q$$

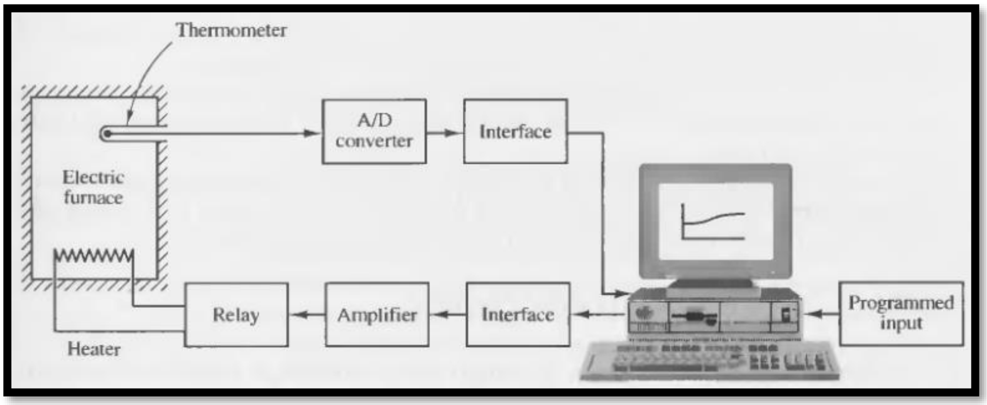
Where c depends on the insulation and the temperature difference between inside and outside chamber



The physical realization of a system to control room temperature is shown below. Here the output signal from a temperature sensing device such as a thermocouple or a resistance thermometer is compared with the desired temperature. Any difference or error causes the controller to send a **control signal** to the gas solenoid valve which produces a linear movement of the valve steam, thus adjusting the flow of gas to the burner of the gas fire. The desired temperature is usually obtained from manual adjustment of a potentiometer

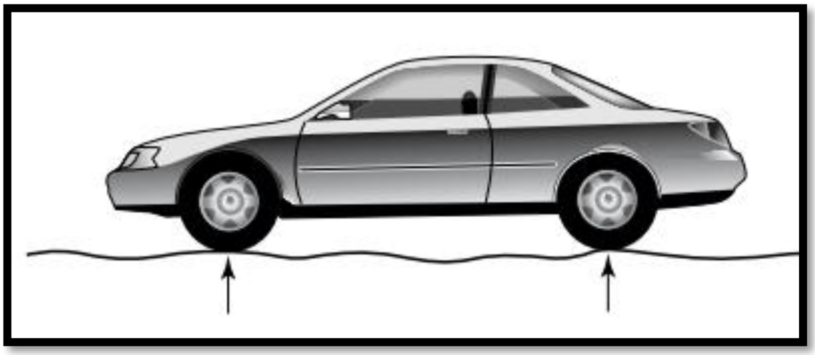


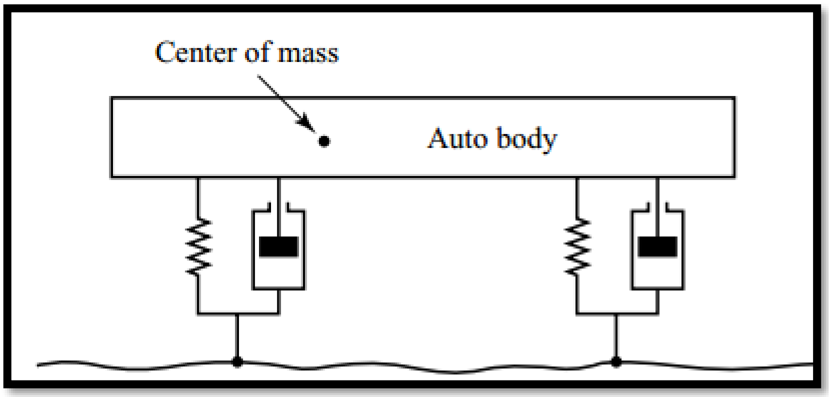




- **Car suspension system**

Figure below shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.





Personal Transporter (PT)

A Segway Personal Transporter (PT) is a two-wheeled vehicle in which the human operator stands vertically on a platform. As the driver leans left, right, forward, or backward, a set of sensitive gyroscopic sensors sense the desired input. These signals are fed to a computer that amplifies them and commands motors to propel the vehicle in the desired direction. One very important feature of the PT is its safety: The system will maintain its vertical position within a specified angle despite road disturbances, such as up hills and down hills or even if the operator over-leans in any direction.

PROCESS-CONTROL BLOCK DIAGRAM

- To provide a practical, working description of process control, it is useful to describe the elements and operations involved in more generic terms. Such a description should be independent of a particular application and thus be applicable to all control situations.
- A model may be constructed using blocks to represent each distinctive element. The characteristics of control operation then may be developed from a consideration of the properties and interfacing of these elements.

Process

- Flow of liquid in and out of the tank, the tank itself, and the liquid all constitute a process to be placed under control with respect to the fluid level.
- In general, a process can consist of a complex assembly of phenomena that relate to some manufacturing sequence.
- Many variables may be involved in such a process, and it may be desirable to control all these variables at the same time.
- There are single variable processes, in which only one variable is to be controlled, as well as multivariable processes, in which many variables, perhaps interrelated,

may require regulation. The **process** is often also called the **plant**

- **Process control**
  - It refers to the methods that are used to control process variables when manufacturing a product. For example, factors such as the proportion of one ingredient to another, the temperature of the materials, how well the ingredients are mixed, and the pressure under which the materials are held can significantly impact the quality of an end product
  - All process systems consist of three main factors or terms: the **manipulated variables**, **disturbances**, and the **controlled variables**.
  - Typical manipulated variables are valve position, motor speed, damper position, or blade pitch.
  - The **controlled variables** are those conditions, such as temperature, level, position, pressure, pH, density, moisture content, weight, and speed, that must be maintained at some desired value.
  - For each controlled variable there is an associated **manipulated variable**. The control system must adjust the manipulated variables so the desired value or “set point” of the controlled variable is maintained despite any disturbances — **The factor that is changed to keep the measured variable at setpoint**

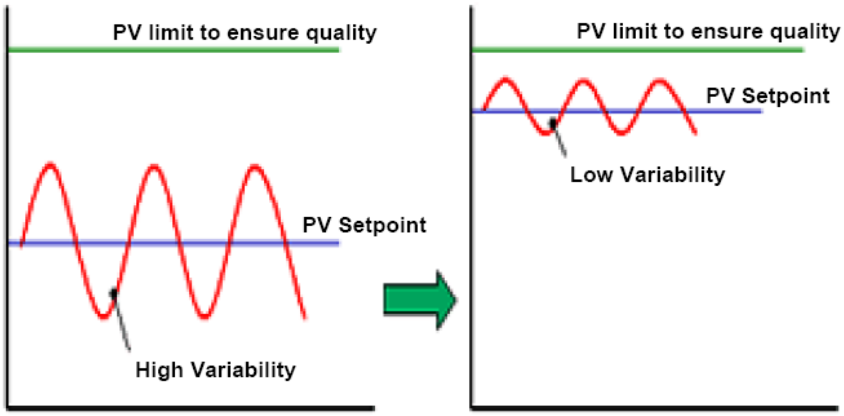
- **Disturbances:** There are other inputs over which the engineer has no control, and these will tend to deflect the plant outputs from their desired values.
- **Process variable:** Temperature – level – pressure
- **Set point: reference signal: desired signal**: The setpoint is a value for a process variable that is desired to be maintained. For example, if a process temperature needs to kept within 5 °C of 100 °C, then the setpoint is 100 °C
- **Actuator:** is the device that can influence the controlled variable of the process

  the central issue with the actuator is its ability to move the process output with adequate speed and range

process + actuator = plant

- **Why to control production process**
  - **Reduce variability**

    gasoline blending process, as many as 12 or more different components may be blended to make a specific grade of gasoline. If the refinery does not have precise control over the flow of the separate components, the gasoline may get too much of the high-octane components. As a result, customers would receive a higher grade and more expensive gasoline than they paid for, and the refinery would lose money. The opposite situation would be customers receiving a lower grade at a higher price



- Increase efficiency
- Ensure safety

- **General Requirements of a Control System**

The primary requirement of a control system is that it be **reasonably stable**. In other words, its speed of **response must be fairly fast**, and this response must show **reasonable damping**. A control system must also be able to **reduce the system error to zero** or to a value near zero

- **System Error**

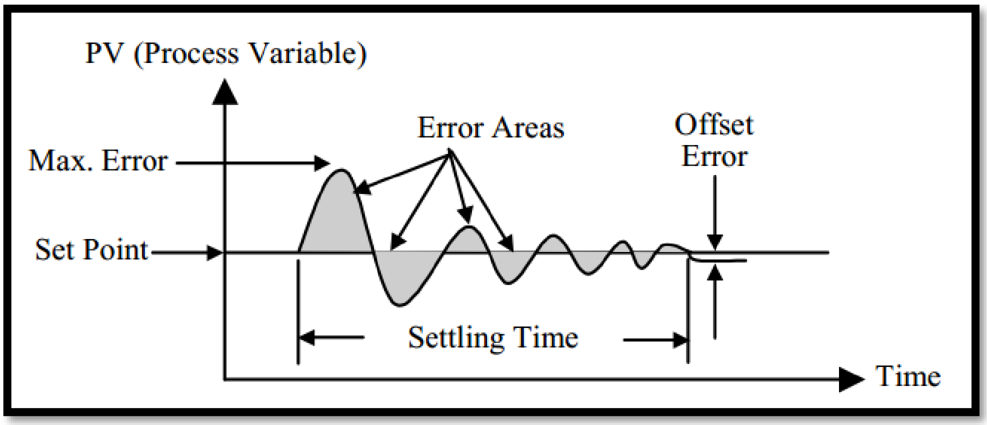
The system error is the **difference** between the value of the **controlled variable** set point and the value of the **process variable** maintained by the system

System Response

The main purpose of a control loop is to maintain some dynamic process variable (pressure, flow, temperature, level, etc.) at a prescribed operating point or set point. System response is the ability of a control loop to recover from a disturbance that causes a change in the controlled process variable

Control Loop Design Criteria

Many criteria are employed to evaluate the process control's loop response to an input change. The most common of these include settling time, maximum error, offset error, and error area



✍ **Note:**

Consider the problem of playing tennis at top international level. One can readily accept that one needs good eye sight (sensors) and strong muscles (actuators) to play tennis at this level, but these attributes are not sufficient. Indeed eye-hand coordination (i.e. control) is also crucial to success

<u>To achieve a good control:</u>

- System must be stable at all times
- System output must track the command input signal
- System output must be prevented from responding too much to disturbance inputs
- These goals must be met even if the model used in the design is not completely accurate or if the dynamic of the physical system change over time or with environmental change

■ OUTLINE OF THE BOOK

In what follows we shall briefly present the arrangements and contents of the book.

- **Chapter 1** has given introductory materials on control systems. **Chapter 2** presents basic Laplace transform theory necessary for understanding the control theory presented in this book. **Chapter 3** deals with mathematical modeling of dynamic systems in terms of transfer functions and state-space equations. **Chapter 4** treats transient-response analyses of first- and second-order systems. This chapter also gives details of transient-response analysis with **MATLAB**. **Chapter 5** first presents basic control actions and then discusses pneumatic, hydraulic, and electronic controllers. This chapter also discusses **Routh's stability** criterion.

- **Chapter 6** gives a root-locus analysis of control systems. General rules for constructing root loci are presented. Detailed discussions for plotting root loci with MATLAB are included.

Chapter 02

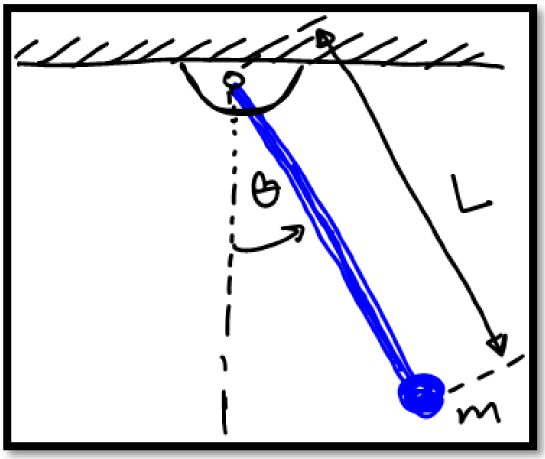
Whole picture Example

Table of content

- Pendulum example
- Model
- Response
- Transfer function
- MATLAB
- SIMULINK

Picture 01

- Consider a simple pendulum having length L, mass m, and instantaneous angular displacement θ, as shown in Figure.



where $\boldsymbol{L = 1\ m, m = 1\ kg}$, and $\boldsymbol{g = 9.8\ m/s^2}$

assume all the mass is concentrated at the end point and there a torque, T_c, applied at the pivot

use MATLAB to determine the time history of θ to a step input in T_c of 1 N.m

$\sum T = I\alpha$

I : moment of inertia = mL^2

Applying Newton 2nd law

$\therefore \quad T_c - mgL\sin\theta = I\ddot{\theta}$

$T_c - mgL\sin\theta = mL^2\ddot{\theta}$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{T_c}{mL^2}$$

This is a nonlinear equation due to $\sin\theta$ term but in small θ, $\sin\theta \approx \theta$

$$\therefore \quad \ddot{\theta} + \frac{g}{L}\theta = \frac{T_c}{mL^2}$$

Taking Laplace

$$\left(s^2 + \frac{g}{L}\right)\theta = \frac{T_c}{mL^2}$$

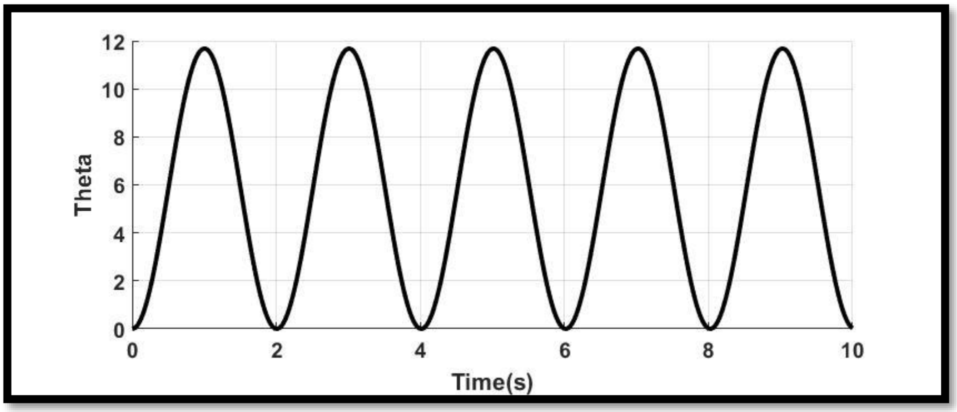
$$\frac{\theta}{T_c} = \frac{1/mL^2}{s^2 + \frac{g}{L}}$$

Substitute with values

$$\frac{\theta}{T_c} = \frac{1}{s^2 + 9.81}$$

MATLAB code: `.m` file

```
a = [1];
b = [1 0 9.81];
sys = tf(a,b);
t = 0:0.02:10;
y = step(sys,t);
plot(t,57.3*y);
```



As we saw, the resulting equation of motion:

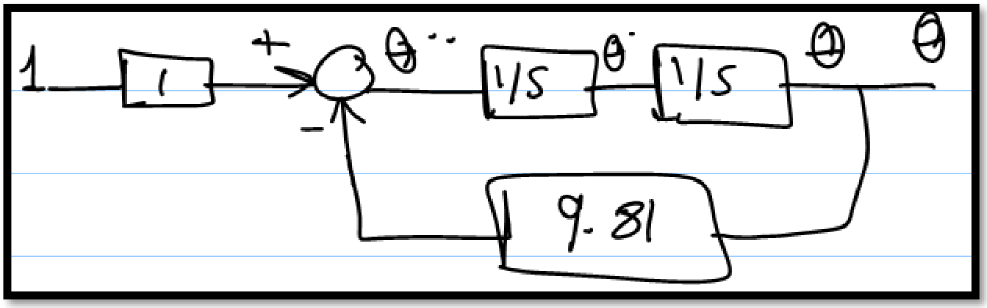
$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{T_c}{mL^2}$$

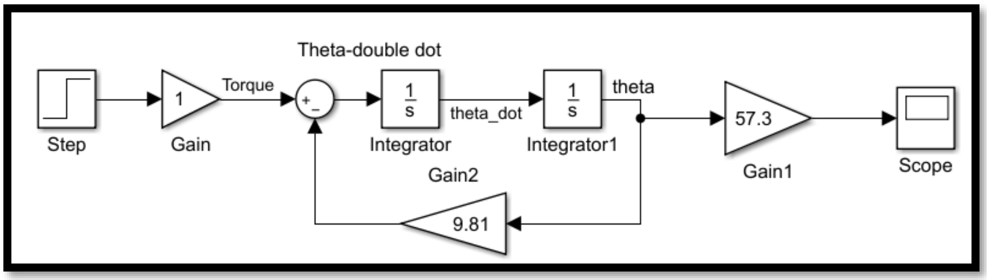
are more difficult to solve than linear ones

it is therefore useful to linearize model in order to gain access to linear analysis methods

$$\ddot{\theta} + \frac{g}{L}\theta = \frac{T_c}{mL^2}$$

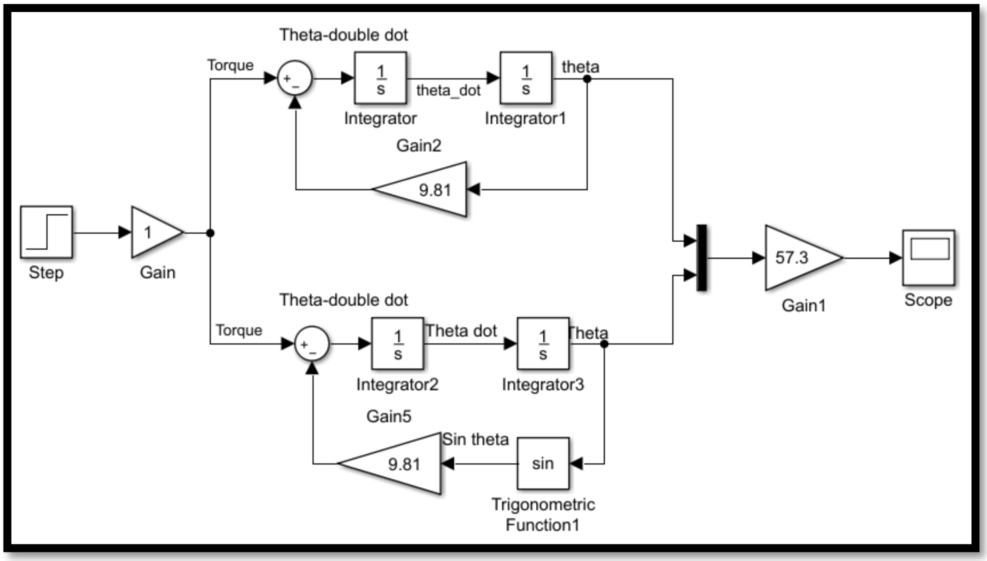
$$\ddot{\theta} = -9.81\,\theta + 1$$





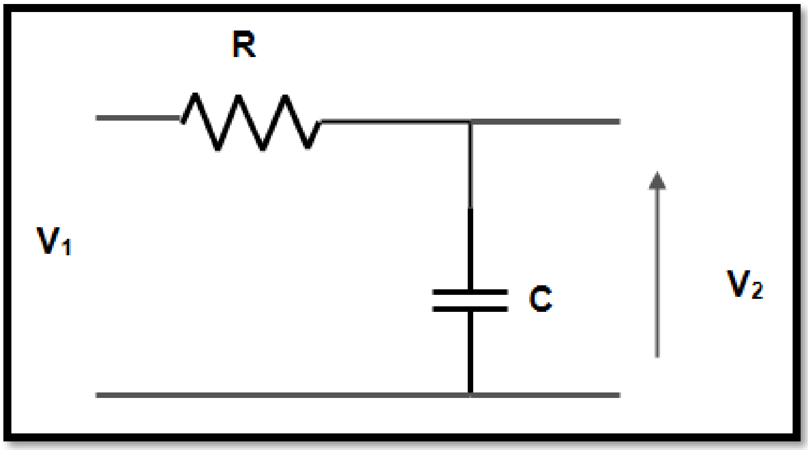
Nonlinear model

$$\ddot{\theta} = -9.81 \sin \theta + 1$$



■ Picture 02:

Find the differential equation relating $\boldsymbol{v_1(t)}$ and $\boldsymbol{v_2(t)}$ for the RC network shown in Figure below



$$v_1(t) = R\,i(t) + \frac{1}{c}\int i(t)\,dt \qquad \text{I}$$

$$v_2(t) = \frac{1}{c}\int i(t)\,dt \qquad \text{II}$$

Taking Laplace

$$V_1 = R\,I + \frac{I}{CS} = I(R + \frac{1}{CS})$$

$$V_2 = \frac{I}{CS}$$

Divide equation I by equation II, then

$$\frac{V_2}{V_1} = \frac{I/CS}{I/(R + \frac{1}{CS})} = \frac{1}{1 + RCS}$$

Assuming voltage source equal to 1 volt is applied the circuit, i.e. $v_1(t) = 1$

$\therefore \quad V_1(t) = 1/s$

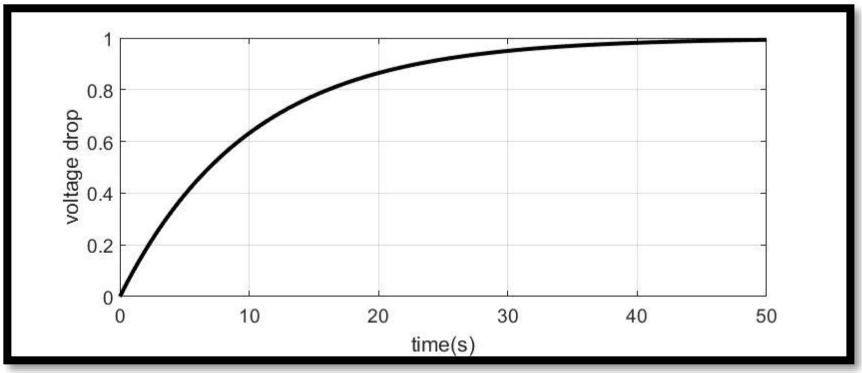
$\therefore \quad V_2 = \frac{1}{s(1+RCS)} = \frac{1}{s} - \frac{RC}{1+RCS} = \frac{1}{s} - \frac{1}{S+1/RC}$

$\therefore \quad v_2(t) = 1 - e^{-t/RC}$

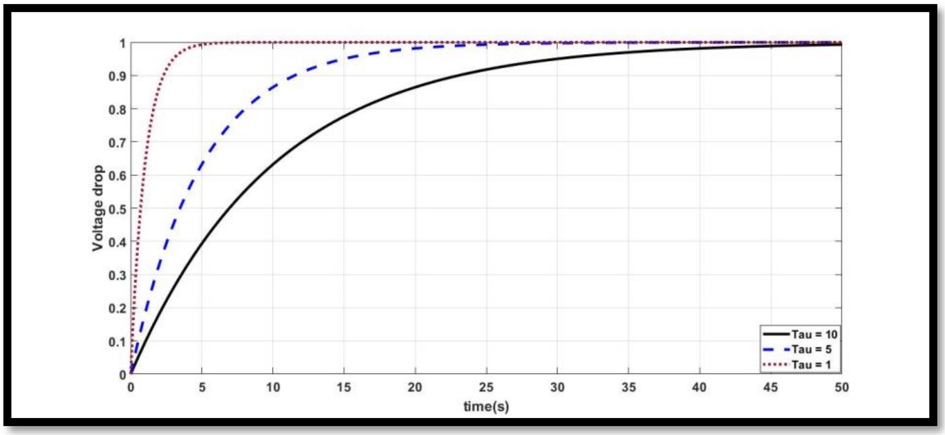
✍ Note:

τ: **time constant** = RC

R and C are **physical parameters**, changing these values will change **time constant**, and finally this will have an impact on the **system response**

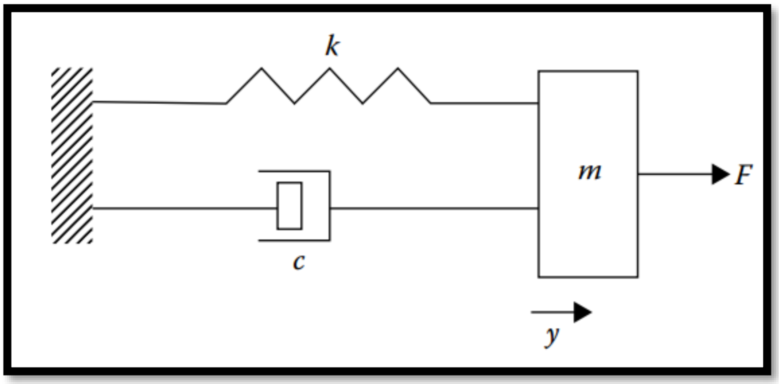


```
a = 1;
b1 = [10 1];
b2 = [5 1];
b3 = [1 1];
sys1 = tf(a,b1);
sys2 = tf(a,b2);
sys3 = tf(a,b3);
t = 0:0.1:50;
y1 = step(sys1,t);
y2 = step(sys2,t);
y3 = step(sys3,t);
plot(t,y1,t,y2,t,y3);
```



■ Picture 03:

Figure below shows a simple mechanical translational system with a **mass**, a **spring**, and a **dashpot**. A force ***F*** is applied to the system. Derive a **mathematical model** for the system.



As shown in figure, the net force on the mass is the applied force minus the forces exerted by the spring and the dashpot. Applying Newton's second law, we can write the system equation as

$$f(t) - c\dot{y}(t) - ky(t) = m\ddot{y}(t)$$

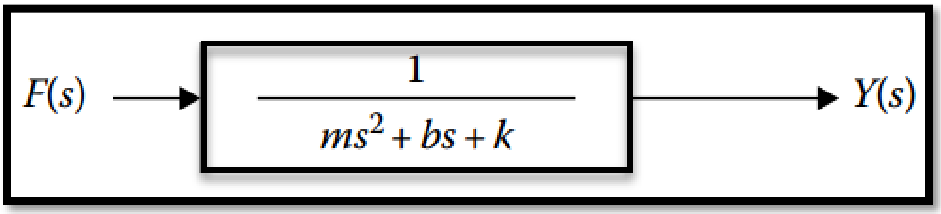
Taking Laplace

$$F(s) - csY(s) - kY(s) = ms^2Y(s)$$

$$Y(s)(ms^2 + cs + k) = F(s)$$

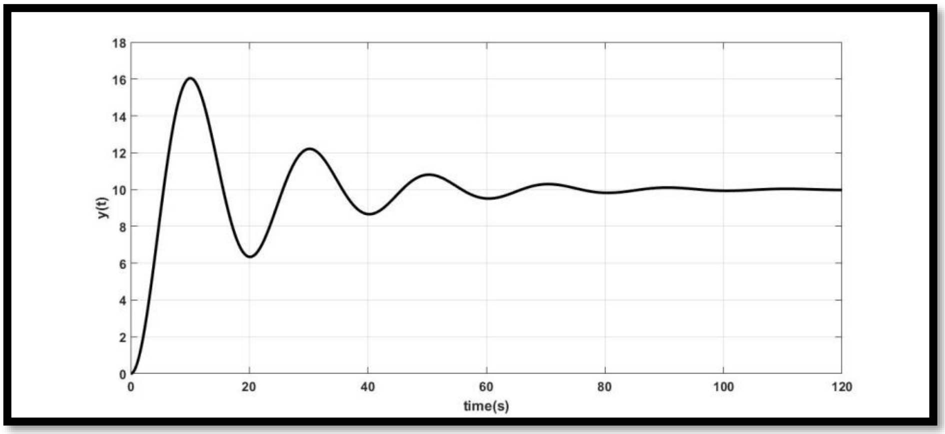
$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2+cs+k}$$ ➔ second-order system

The **transfer function** in Equation above is represented by the **block diagram** shown in Figure below.

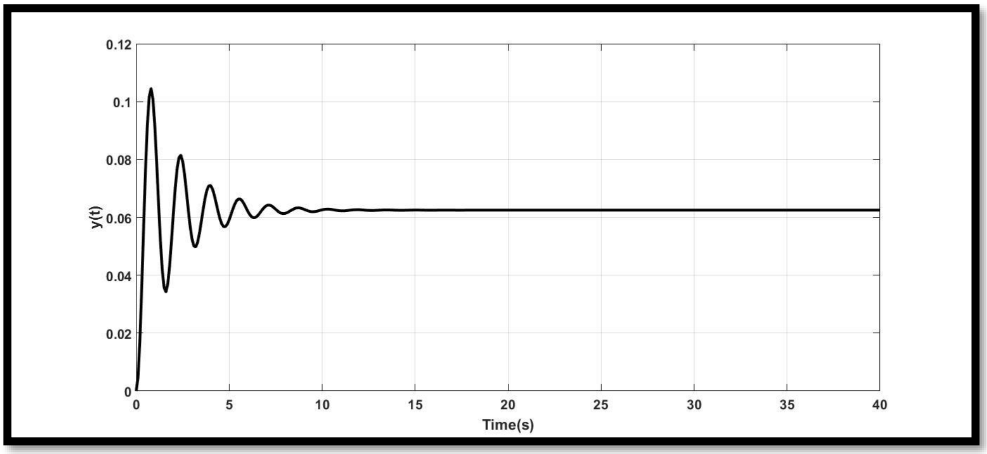


The **step response** of this mechanical system may be determined by simulating the following **MATLAB codes** and the response obtained is shown in Figure below.

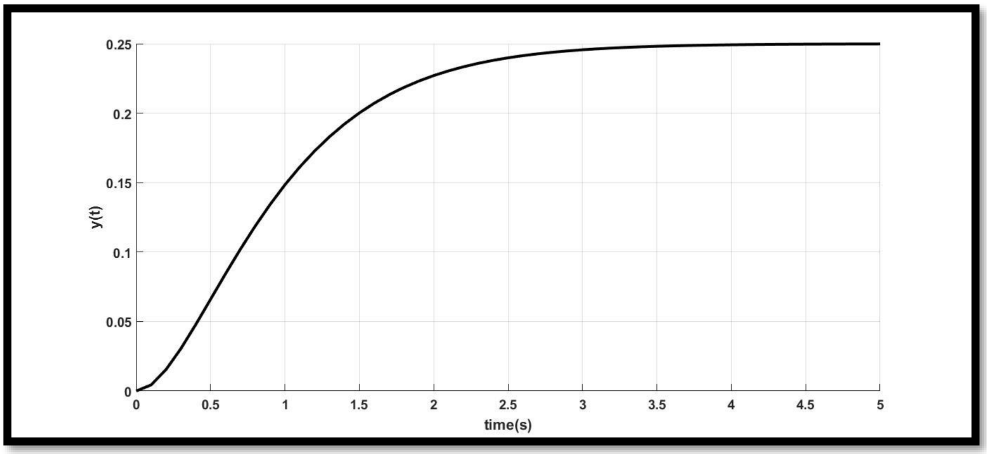
```
m=1.0; % kg
c=0.1; %
k=0.1; %
num = [1];
den=[m c k];
sys = tf(num,den);
t = 0:0.1:120;
y = step(sys,t);
plot(t,y);
```



```
m=1.0; % kg
c=1; %
k=16; %
num = [1];
den=[m c k];
sys = tf(num,den);
t = 0:0.1:120;
y = step(sys,t);
plot(t,y);
```




```
m=1.0; % kg
c=4; %
k=4; %
num = [1];
den=[m c k];
sys = tf(num,den);
t = 0:0.1:120;
y = step(sys,t);
plot(t,y);
```


y(t)
time(s)
0
0.05
0.1
0.15
0.2
0.25
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Chapter 03

Laplace Transform

Table of content

- Laplace transforms
- Partial fraction
- Differential equation
- Initial value theorem
- Final value theorm
- MATLAB
- SIMULINK

■ Laplace transforms

- $F(s) = \mathcal{L}\, f(t) = \int_0^\infty e^{-st} f(t)\, dt$
- This technique transforms the problem from the time (t) domain to the Laplace (s) domain. The advantage in doing this is that complex time domain differential equations become relatively simple s domain algebraic equations. When a suitable solution is arrived at, it is inverse transformed back to the time domain.
- where s is a complex variable $\sigma \pm j\omega$ and ℓ is called the Laplace operator

Example

Unit-step input $f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases}$

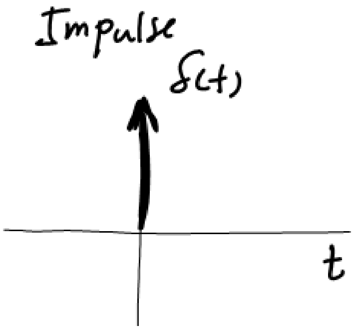
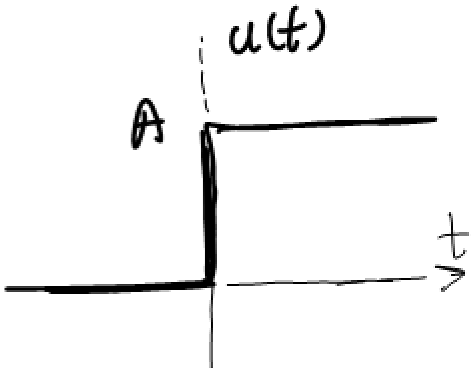
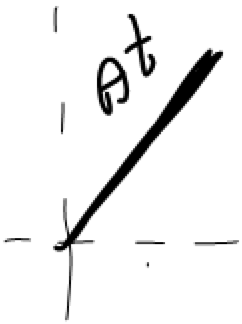
$$F(s) = \int_0^\infty e^{-st}\, 1\, dt = \left. \frac{e^{-st}}{-s} \right|_\infty^0 = 1/s$$

Example

Unit-step input $f(t) = \begin{cases} a & t \geq 0 \\ 0 & otherwise \end{cases}$

$$F(s) = \int_0^\infty e^{-st}\, a\, dt = a \left. \frac{e^{-st}}{-s} \right|_\infty^0 = a/s$$

- **Test functions**

Impulse $\delta(t)$	
Unit-step $u(t)$	
Ramp $f(t) = At$	

Common Laplace transform pairs

	Time function $f(t)$	Laplace transform $\mathscr{L}[f(t)] = F(s)$
1	unit impulse $\delta(t)$	1
2	unit step 1	$1/s$
3	unit ramp t	$1/s^2$
4	t^n	$\frac{n!}{s^{n+1}}$
5	e^{-at}	$\frac{1}{(s+a)}$
6	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
7	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
8	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
9	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
10	$e^{-at}(\cos \omega t - \frac{a}{\omega} \sin \omega t)$	$\frac{s}{(s+a)^2+\omega^2}$

Important properties of Laplace transform

- **Constant multiplication**

$$\mathcal{L}\,[af(t)] = a\,F(s)$$

- **Real-shift theorem**

$$\mathcal{L}\,f(t-T) = e^{-sT}F(s) \quad T \geq 0$$

- **Convolution Integral**

$$\int_0^t f_1(\tau) f_2(t-\tau) \quad d\tau = F_1(s)F_2(s)$$

- **Initial value theorem**

$$f(0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,F(s)$$

- **Final value theorem**

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

- **Real differentiation theorem**

$$\mathcal{L}\,\frac{d^n}{dt^n} f(t) = s^n F(s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0) - s^{n-3} \ddot{f}(0) \ldots f(0)^{n-1}$$

- **Integration**

$$\mathcal{L} \int_0^t f(\tau)\, d\tau = \frac{F(s)}{s}$$

- **Multiplication by t^n**

$$\mathcal{L}[t^n f(t)] = -1^n \frac{d^n}{ds^n} F(s) \quad for\ n = 1,2,3, \ldots$$

Common partial fraction expansion

- Factored roots

$$\frac{k}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

- Repeated roots

$$\frac{k}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

- Second-order real roots $b^2 > 4ac$

$$\frac{K}{s(as^2 + bs + c)} = \frac{K}{s(s+d)(s+e)} = \frac{A}{s} + \frac{B}{s+d} + \frac{C}{s+e}$$

- Second-order real roots $b^2 < 4ac$

$$\frac{K}{s(as^2 + bs + c)} = \frac{A}{s} + \frac{Bs + C}{as^2 + bs + c}$$

Completing the squares, gives

$$\frac{A}{s}+\frac{Bs+C}{(s+\alpha)^2+\omega^2}$$

$f(t)$	$F(s)$
Unit impulse $\delta(t)$	1
Unit step $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!} \quad (n=1,2,3,\ldots)$	$\frac{1}{s^n}$
$t^n \quad (n=1,2,3,\ldots)$	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at} \quad (n=1,2,3,\ldots)$	$\frac{1}{(s+a)^n}$
$t^n e^{-at} \quad (n=1,2,3,\ldots)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
$1-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
$\frac{1}{2\omega} t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

Find Laplace transform of

$$f(t) = e^{-at} \sin \omega t$$

$\because \quad \sin \omega t = \dfrac{\omega}{s^2 + \omega^2}$

$\therefore \quad F(s) = \dfrac{\omega}{(s+a)^2 + \omega^2}$

Given that:

$$Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Find $y(t)$

By partial fraction

$$Y(s) = \frac{8/3}{s} - \frac{3/2}{s+1} - \frac{1/6}{s+3}$$

$$y(t) = \frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Find the inverse Laplace of

$$\frac{s+3}{(s+1)(s+2)}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} + \frac{-1}{(s+2)}$$

$\ell^{-1} = 2e^{-t} - e^{-2t}$

Find the inverse Laplace of

$$F(s) = \frac{2s+12}{s^2+2s+5}$$

$$F(s) = \frac{2(s+1)+10}{(s+1)^2+4}$$

$$F(s) = 2\frac{(s+1)}{(s+1)^2+4} + 5\frac{2}{(s+1)^2+4}$$

$$f(t) = 2e^{-t}\cos 2t + 5e^{-t}\sin 2t$$

Find the solution to the differential equation

$\ddot{y}(t) + y(t) = 0$ where $y(0) = \alpha$, $\dot{y}(0) = \beta$

$$s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = 0$$

$$Y(s)[s^2+1] = \alpha s + \beta$$

$$Y(s) = \frac{\alpha s + \beta}{s^2+1}$$

$$y(t) = \alpha\cos t + \beta\sin t$$

Find the solution to the differential equation

$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = u(t)$ where $y(0) = 0$, $\dot{y}(0) = 0$

$u(t) = 2e^{-2t}$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 5[sY(s) - y(0)] + 4Y(s) = \frac{2}{s+2}$$

$$Y(s)[s^2 + 5s + 4] = \frac{2}{s+2}$$

$$Y(s) = \frac{2}{(s+2)(s^2+5s+4)} = \frac{2}{(s+2)(s+1)(s+4)}$$

$$Y(s) = \frac{2/3}{s+1} - \frac{1}{s+2} + \frac{1/3}{s+4}$$

$$y(t) = \frac{2}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t}$$

Find inverse Laplace of

$$Y(s) = \frac{s+2}{s^3(s-1)^2}$$

$$\frac{s+2}{s^3(s-1)^2} = \frac{2}{s^3} + \frac{A}{s^2} + \frac{B}{s} + \frac{3}{(s-1)^2} + \frac{C}{s-1}$$

$$\therefore \quad \frac{s+2}{s^3(s-1)^2} = \frac{2}{s^3} + \frac{5}{s^2} + \frac{8}{s} + \frac{3}{(s-1)^2} - \frac{8}{s-1}$$

$$\therefore \quad y(t) = t^2 + 5t + 8 + 3te^t - 8e^t$$

Find inverse Laplace of

$$Y(s) = \frac{3s^2 + 9s + 16}{(s+1)(s^2 + 4s + 13)}$$

$$Y(s) = \frac{1}{s+1} + \frac{As + B}{s^2 + 4s + 13}$$

$$Y(s) = \frac{1}{s+1} + \frac{2s + 3}{s^2 + 4s + 13}$$

$$Y(s) = \frac{1}{s+1} + \frac{2(s+2) - 1}{(s+2)^2 + 9}$$

$$Y(s) = \frac{1}{s+1} + 2\frac{(s+2)}{(s+2)^2 + 9} + \frac{-1}{(s+2)^2 + 9}$$

$$y(t) = e^{-t} + 2e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$$

Given that:

$$\dddot{y}(t) - 6\ddot{y}(t) + 11\dot{y}(t) - 6y(t) = 1$$

$$y(0) = \dot{y}(0) = \ddot{y}(0) = 0$$

Ans.

$$y(t) = -\frac{1}{6} + \frac{1}{2}e^t - \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t}$$

Given that: $\ddot{y}(t) + 2\dot{y}(t) + y(t) = t$ $y(0) = -3 \qquad y(1) = -1$
Ans. Assume $\dot{y}(0) = A$ $y(t) = (t-1)e^{-t} + t - 2$

Given that: $\ddot{x}(t) + 4x(t) = 10\sin 3t$ $x(0) = \dot{x}(0) = 0$

$$s^2X(s) - sx(0) - \dot{x}(0) + 4X(s) = \frac{30}{s^2+9}$$

$$X(s)(s^2+4) = \frac{30}{s^2+9}$$

$$X(s) = \frac{30}{(s^2+4)(s^2+9)}$$

$$X(s) = \frac{6}{(s^2+4)} - \frac{6}{(s^2+9)}$$

$$x(t) = 3\sin 2t - 2\sin 3t$$

Given that:

$\dddot{y}(t) + y(t) = 1$

$y(0) = \dot{y}(0) = \ddot{y}(0) = 0$

Find $y(t)$

- Final value theorem

  Suppose we have a transform $Y(s)$ of a signal $y(t)$ and wish to know the final value $y(t)$ from $Y(s)$: there are three possibilities

  1. $Y(s)$ has any **poles** in the **RHS** (Right hand side) — that is, if real part of any $P_i > 0$ then, $y(t)$ will grow and the limit will be unbounded
  2. $Y(s)$ has a pair of poles on the imaginary axis of the s-plane (i.e. $P_i = \pm j\omega$) then, $y(t)$ will have a sinusoid that persist forever and the final value will not be defined
  3. All poles of $Y(s)$ are in LHS (Left Hand Side), except for one at $s = 0$, then all terms of $y(t)$ will decay to zero except the term corresponding to the pole at $s = 0$, and that term corresponds to a constant in time

Find the final value of the system corresponds to

$$Y(s) = \frac{3(s+2)}{s(s^2+2s+10)}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$\lim_{t\to\infty} y(t) = 6/10$$

Thus, after the transients have decayed to zero, $y(t)$ will settle to a constant value of 0.6

Note:

Care must be taken to use final value theorem in an unstable system

Find the final value of the system corresponds to

$$Y(s) = \frac{3}{s(s-2)}$$

If we blindly apply the equation of final value theorem, we obtain

$y(\infty) = -\frac{3}{2}$

However:

$y(t) = -\frac{3}{2} + \frac{3}{2}e^{2t}$ — unbounded

■ MATLAB session

Residue

- Partial fraction expansion
- **`[r,p,k] = residue(b,a)`** finds the **residues**, **poles**, and **direct term** of a Partial Fraction Expansion of the ratio of two polynomials, where the expansion is of the form

$$\frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{r_n}{s-p_n} + \ldots + \frac{r_2}{s-p_2} + \frac{r_1}{s-p_1} + k(s).$$

- The inputs to **residue** are vectors of coefficients of the polynomials **`b = [bm ... b1 b0]`** and **`a = [an ... a1 a0]`**. The outputs are the residues `r = [rn ... r2 r1]`, the poles **`p = [pn ... p2 p1]`**, and the polynomial `k`. For most textbook problems, `k` is 0 or a constant.
- `[b,a] = residue(r,p,k)` converts the partial fraction expansion back to the ratio of two polynomials and returns the coefficients in b and a.

> Find the partial fraction expansion of the following ratio of polynomials $F(s)$ using residue
>
> $$F(s) = \frac{b(s)}{a(s)} = \frac{-4s+8}{s^2+6s+8}$$

```
b = [-4 8]
a = [1 6 8]
[r,p,k] = residue(b,a)
```

```
r =

   -12
     8


p =

    -4
    -2


k =

     []
```

This represents the partial fraction expansion

$$\frac{-4s+8}{s^2+6s+8} = \frac{-12}{s+4} + \frac{8}{s+2}$$

Convert the partial fraction expansion back to polynomial coefficients using `residue`.

```
[b,a] = residue(r,p,k)
```

```
b =

    -4     8

a =
```

1 6 8

This result represents the original fraction $F(s)$.

Find partial fraction expansion of

$$Y(s) = \frac{s+2}{s^3(s-1)^2}$$

Hint: use **conv** → Convolution and polynomial multiplication

```
b = [1 2];
a1 = [1 0 0 0];
a2 = [1 -1];
a3 = [1 -1];
a = conv(a1,a2);
a = conv(a,a3);
[r,p,k] = residue(b,a)
```

r =

-8

3

8

5

2

p =

1

1

0

0

0

k =

[]

$$\frac{s+2}{s^3(s-1)^2} = \frac{-8}{s-1} + \frac{3}{(s-1)^2} + \frac{8}{s} + \frac{5}{s^2} + \frac{2}{s^3}$$

Find the partial fraction expansion of

$$Y(s) = \frac{s^2+3}{(s+1)(s-2)(s-4)}$$

Hint: use **conv** → Convolution and polynomial multiplication

```
b = [1 0 3];
a1 = [1 1];
a2 = [1 -2];
a3 = [1 -4];
a = conv(a1,a2);
a = conv(a,a3);
[r,p,k] = residue(b,a)
```

r =

1.9000

-1.1667

0.2667

p =

4.0000

2.0000

-1.0000

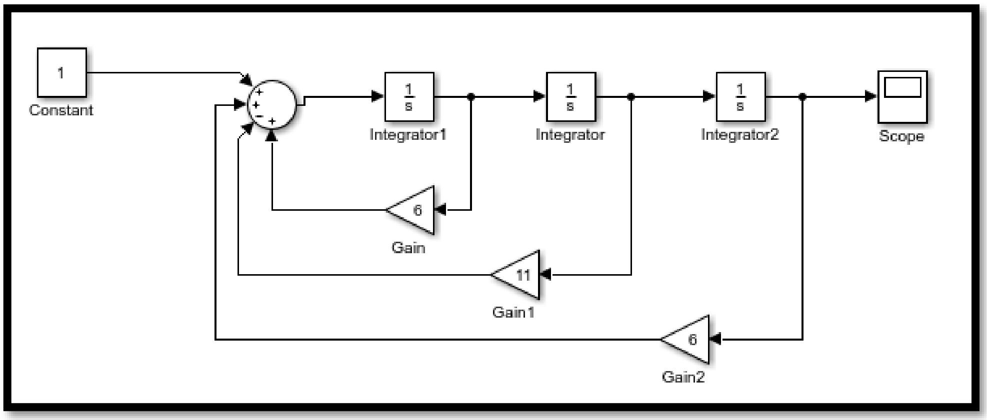
k =

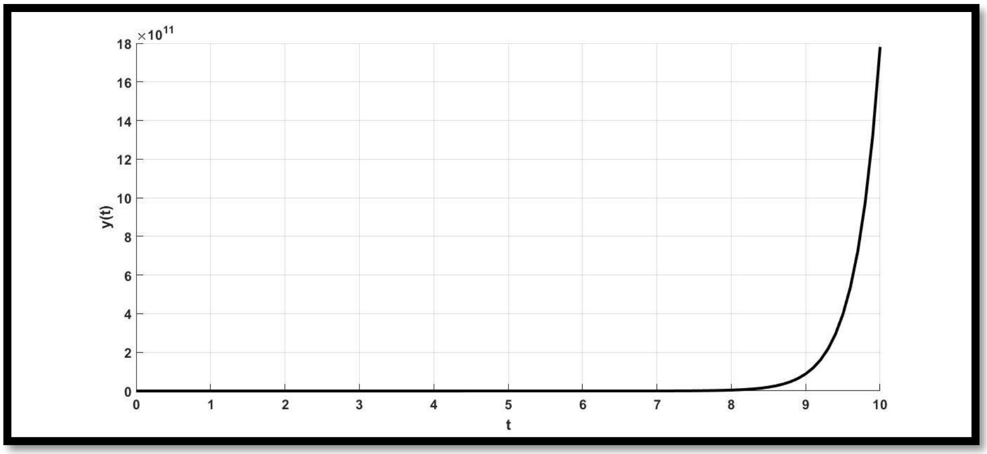
[]

$$\frac{s^2+3}{(s+1)(s-2)(s-4)} = \frac{1.9}{s-4} - \frac{1.1667}{s-2} + \frac{0.226}{s+1}$$

Given that: $\dddot{y}(t) - 6\ddot{y}(t) + 11\dot{y}(t) - 6y(t) = 1$ $y(0) = \dot{y}(0) = \ddot{y}(0) = 0$ Find $y(t)$ by MATLAB and SIMULNIK
Ans. $y(t) = -\frac{1}{6} + \frac{1}{2}e^{t} - \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t}$

```
t = 0:0.1:10;
y = -1/6+0.5*exp(t)-0.5*exp(2*t)+(1/6)*exp(3*t);
plot(t,y)
```

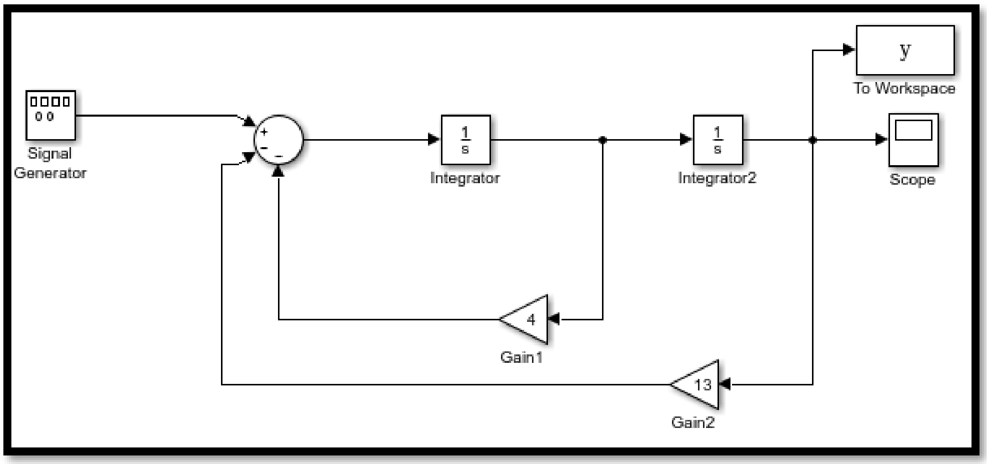


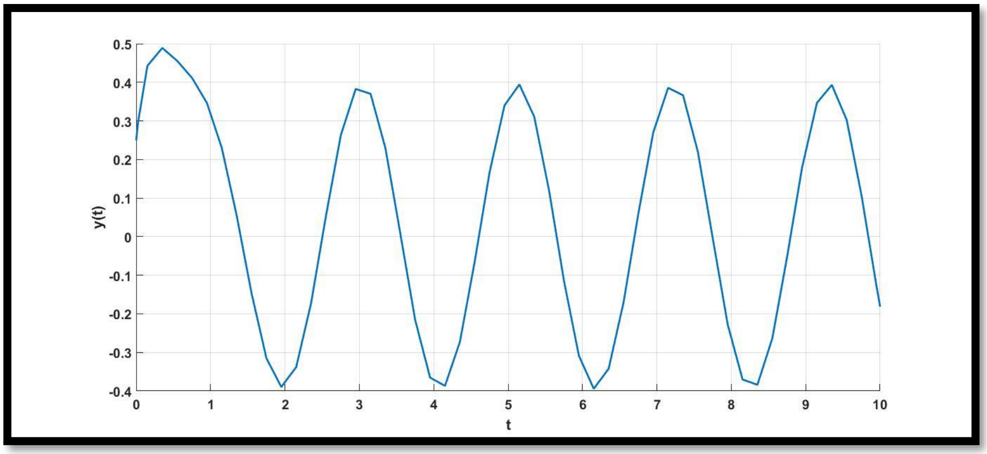


Solve the equation

$$\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 5\cos 3t$$

Given that $y(0) = \frac{1}{4}$, $\dot{y}(0) = 2$





Problems:

1. Solve the equation: $\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 5\cos 3t$

   Given that $y(0) = \frac{1}{4}$, $\dot{y}(0) = 2$

   This is an example of ***forced oscillations with damping***—

   Sketch the Plot of $y(t)$

2. Solve the equation: $\ddot{x}(t) + 2\dot{x}(t) + x(t) = 3te^{-t}$

   Given that $x(0) = 4$, $\dot{x}(0) = 2$

Chapter 04

Modeling

Table of content

- Mathematical model
- Mechanical systems
- Electrical systems
  - RLC circuit
  - OP-AMP
- Electromechanical systems
  - Motor
    - Field control
    - Armature control
- MATLAB session
- SIMULINK

■ Mathematical model

- If the dynamic behavior of a physical system can be represented by an equation, or a **set of equations**, this is referred to as the mathematical model of the system.
    - Mathematical description of the process to be controlled
    - A set of differential equations that describe the dynamic behavior of the process
- Such models can be constructed from **knowledge of the physical characteristics of the system**, i.e. **mass** for a mechanical system or **resistance** for an electrical system.
- Alternatively, a mathematical model may be determined by **experimentation**, by measuring **how the system output responds to known inputs**.
- In general, consider a system whose output is $\boldsymbol{y(t)}$, whose input is $\boldsymbol{x(t)}$and contains constant coefficients of values a, b, c. If the dynamics of the system produce a **first-order** differential equation, it would be represented as

$$\boldsymbol{a\frac{dy(t)}{dt} + by(t) = cx(t)}$$

- If the system dynamics produced a **second-order** differential equation, it would be represented by

$$\boldsymbol{a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = ex(t)}$$

- If the dynamics produce a **third-order** differential equation, its representation would be

$$\boldsymbol{a\frac{d^3y(t)}{dt^3}+b\frac{d^2y(t)}{dt^2}+c\frac{dy(t)}{dt}+ey(t)=fx(t)}$$

- Equations above are linear differential equations with constant coefficients. Note that the **order of the differential equation** is the **order of the highest derivative**.
- Systems described by such equations are called **linear systems** of the same order as the differential equation. For example, **first-order linear system**, **second-order** linear system and **third-order** linear system.

A control system may be composed of various components including **mechanical**, **thermal**, **fluid**, **pneumatic**, and **electrical**; **sensors** and **actuators**; and **computers**

Using the basic **modeling principles** such as **Newton's second law** of motion or **Kirchhoff's law**, the models of these dynamic systems are represented by **differential equations**. It is not difficult to understand that the analytical and computer simulation of any system is only as good as the model used to describe it. It should also be emphasized that the modern control engineer should place special emphasis on the mathematical modeling of systems so that analysis and design problems can be conveniently solved by computers.

■ Mechanical systems modeling

Mechanical systems may be modeled as **systems of lumped masses** (**rigid bodies**) or as **distributed mass** (continuous) systems. The latter are modeled by **partial differential equations**, whereas the former is represented by **ordinary differential equations**.

Of course, in reality all systems are continuous, but, in most cases, it is easier and therefore preferred to approximate them with lumped mass models and ordinary differential equations

The motion of mechanical elements can be described in various dimensions as **translational**, **rotational**, or a **combination of both**.

The equations governing the motion of mechanical systems are often directly or indirectly formulated from Newton's law of motion.

The equations of a linear mechanical system are written by first constructing a model of the system containing interconnected linear elements and then by applying **Newton's law of motion** to the **free-body diagram** (FBD).

- **Translational motion**

The motion of **translation** is defined as a **motion that takes place along a straight or curved path**. The variables that are used to describe translational motion are **acceleration**, **velocity**, and **displacement**.

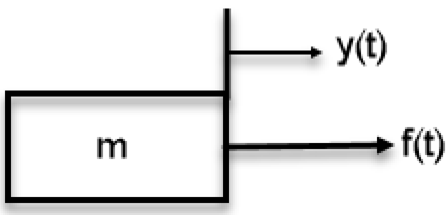
> **Newton's law of motion** states that the algebraic sum of external forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction.

The law can be expressed as

$$\sum_{External} \boldsymbol{Forces} = \boldsymbol{m\,a}$$

where $\boldsymbol{m}$ denotes the mass, and $\boldsymbol{a}$ is the acceleration in the direction considered.

Fig. below illustrates the situation where a force is acting on a body with mass $\boldsymbol{m}$ The force equation is written as



$$f(t) = m\,a(t) = m\frac{d^2y}{dt^2} = m\frac{dv}{dt}$$

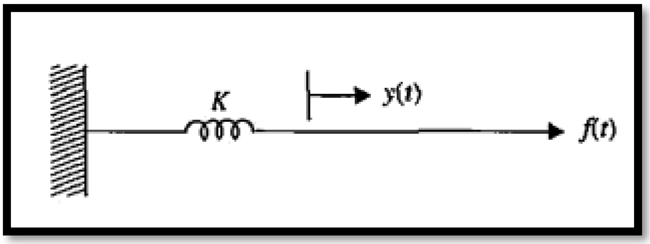
where $\boldsymbol{a}(\boldsymbol{t})$ is the acceleration, $\boldsymbol{v}(\boldsymbol{t})$ denotes linear velocity, and $y(t)$ is the displacement of mass $\boldsymbol{m}$, respectively. For linear translational motion, in addition to the mass, the following system elements are also involved.

Linear spring. In practice, a linear spring may be a model of an actual spring or a compliance of a cable or a belt. In general, a spring is considered to be an element that stores potential energy.

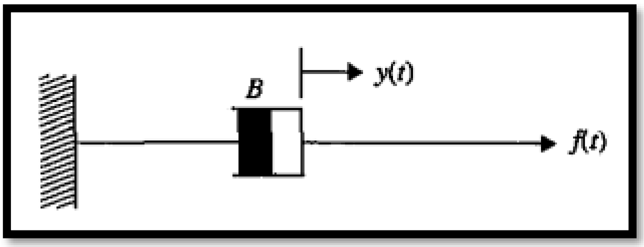
$$f(t) = k\, y(t)$$

where $\boldsymbol{k}$ is the spring constant, or simply stiffness. Equation above implies that the **force acting on the spring is directly proportional to the displacement (deformation) of the spring**.

The model representing a linear spring element is shown in Fig. below.



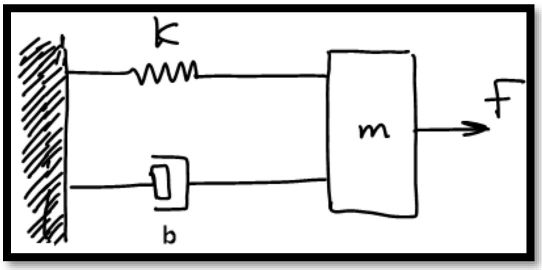
Friction: Whenever there is motion or tendency of motion between two physical elements, frictional forces exist, i.e. that is a **linear relationship between the applied force and velocity**

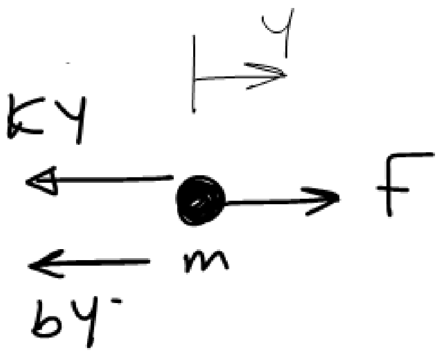


$$f(t) = \beta \frac{dy}{dt}$$

where β is the viscous frictional coefficient

Consider the mass-spring-friction system shown in Fig. below. The linear motion concerned is in the horizontal direction.





$f(t) - b\dot{y}(t) - ky(t) = m\ddot{y}(t)$

$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = f(t)$

Taking Laplace, it gives, and assume zero initial condition

$Y(s)[ms^2 + bs + k] = F(s)$

$\dfrac{\boldsymbol{Y(s)}}{\boldsymbol{F(s)}} = \dfrac{\mathbf{1}}{\boldsymbol{ms^2 + bs + k}}$	Transfer function
▪ $\boldsymbol{m}$, $\boldsymbol{b}$, $\boldsymbol{k}$ are physical parameters ▪ Changing these parameters, also change the **system response**	

Assume $f(t) = u(t) = 1\ N$

$$\therefore \quad F(s) = \frac{1}{s}$$

And $m = 1$, $b = 0.1$, $k = 0.1$

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{F(s)}} = \frac{\mathbf{1}}{\boldsymbol{s^2 + 0.1s + 0.1}}$$

1. **Analytically**

$$Y(s) = \frac{1}{s(s^2 + 0.1s + 0.1)} = \frac{10}{s} + \frac{As + B}{s^2 + 0.1s + 0.1}$$

$$\therefore \quad Y(s) = \frac{10}{s} - \frac{10s + 1}{s^2 + 0.1s + 0.1}$$

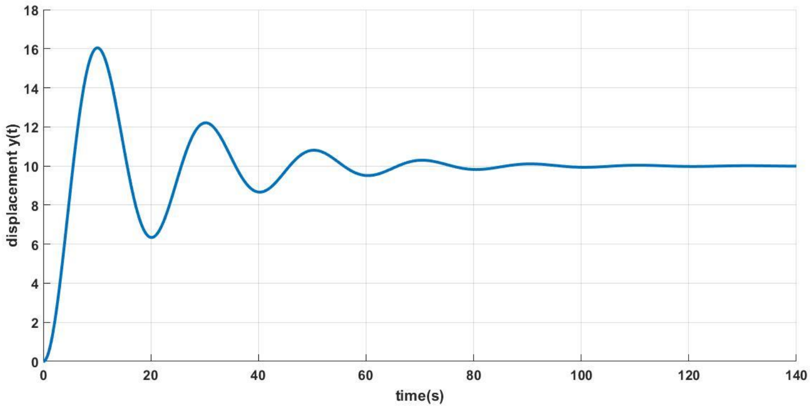
$$Y(s) = \frac{10}{s} - \frac{10(s + 0.05) + 0.5}{(s + 0.05)^2 + 0.0975}$$

$$y(t) = 10 - 10e^{-0.05t}\cos\sqrt{0.0975}t - \frac{0.5}{\sqrt{0.0975}}e^{-0.05t}\sin\sqrt{0.0975}t$$

2. By MATLAB control system toolbox

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{F(s)}} = \frac{\mathbf{1}}{\boldsymbol{s^2 + 0.1s + 0.1}}$$

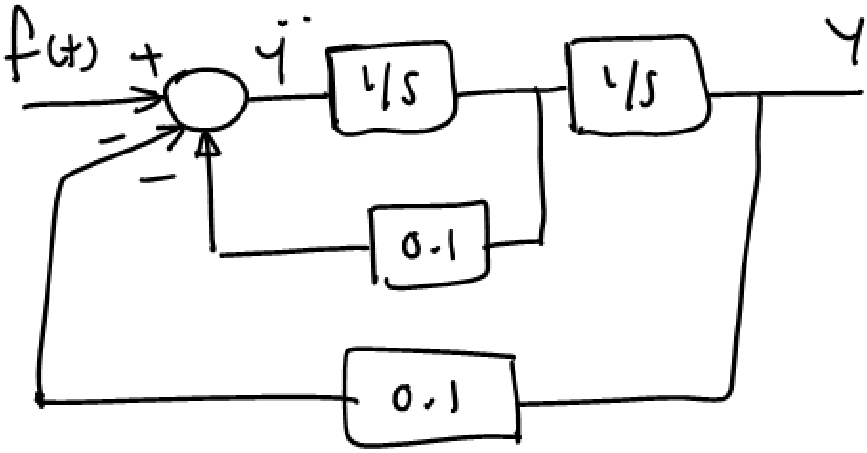
```
b = 0.1;
k = 0.1;
num = [1];
den = [m b k];
t = 0:0.1:140;
sys = tf(num,den);
y = step(sys,t);
plot(t,y);
```

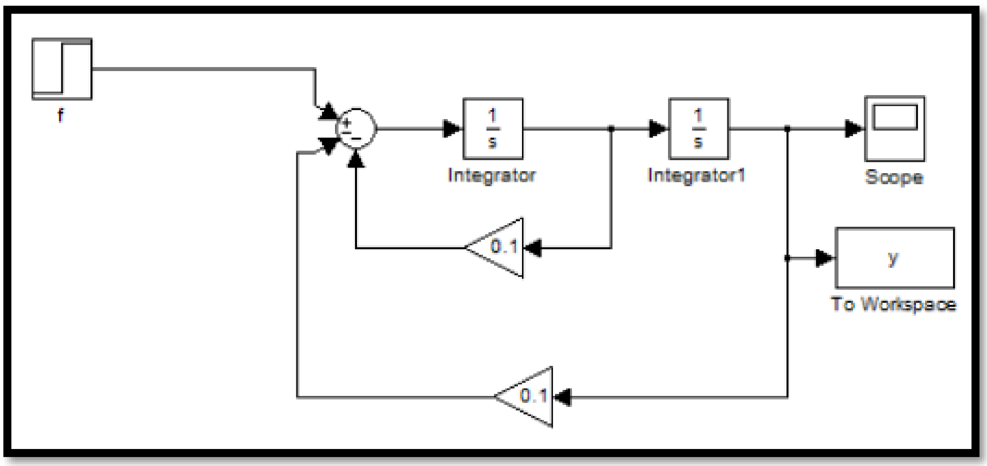


3. By Simulink

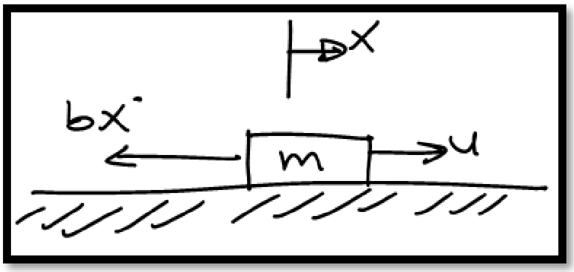
$$\frac{Y(s)}{F(s)} = \frac{1}{s^2 + 0.1s + 0.1}$$

$$\ddot{y} + 0.1\dot{y} + 0.1y = f(t)$$





Write the equation of motion for the speed and forward motion of the car, assuming that the engine impacts a force u as shown, find the transfer function between the input u and the output v



$u - b\dot{x} = m\ddot{x}$

$m\ddot{x} + b\dot{x} = u$

$m\dot{v} + bv = u$

Take Laplace transform

$(ms + b)V = U$

$$\frac{V}{U} = \frac{1}{ms + b} = \frac{1/m}{s + b/m}$$

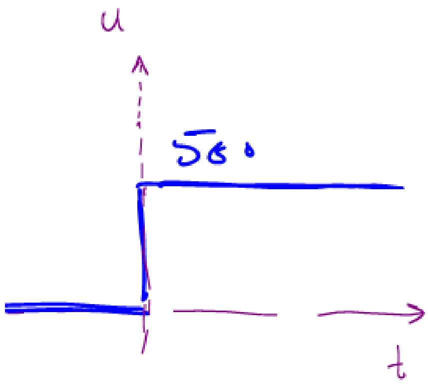
Example:

Use MATLAB to find the response of the velocity of the car for the case in which the input jumps from being $u = 0$ at time $t = 0$ to a constant $u = 500\ N$ thereafter. Assume that the car mass is 1000 kg and viscous drag coefficient $b = 50$

$$\because \quad \frac{V}{U} = \frac{1/m}{s + b/m} = \frac{1/1000}{s + 50/1000}$$

$$\frac{V}{U} = \frac{0.001}{s + 0.05}$$

$$u = \begin{cases} 500 & t \geq 0 \\ 0 & otherwise \end{cases}$$



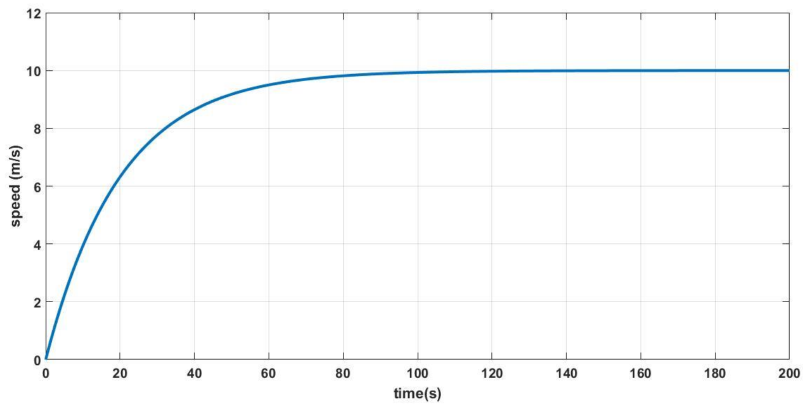
$$U(s) = 500/s$$

$$\therefore \quad V(s) = \frac{0.5}{s(s + 0.05)}$$

Partial fraction

$$V(s) = \frac{10}{s} - \frac{10}{s + 0.05}$$

$$v(t) = 10 - 10e^{-0.05t}$$

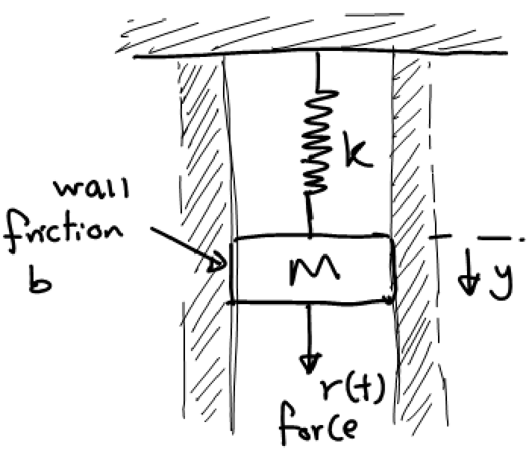


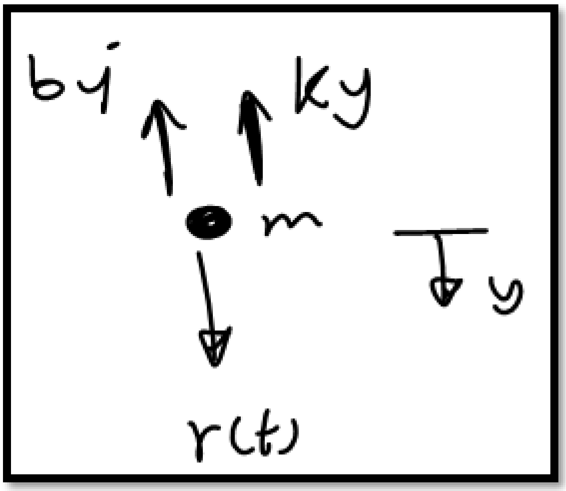
You can also plot using MATLAB toolbox

$$\frac{V}{U} = \frac{0.001}{s + 0.05}$$

```
num = [0.001];
den = [1 0.05];
sys = tf(num,den);
step(500*sys)
```

Example: Spring-mass-damper (automobile shock absorber)





$$m\ddot{y} = r - b\dot{y} - ky$$

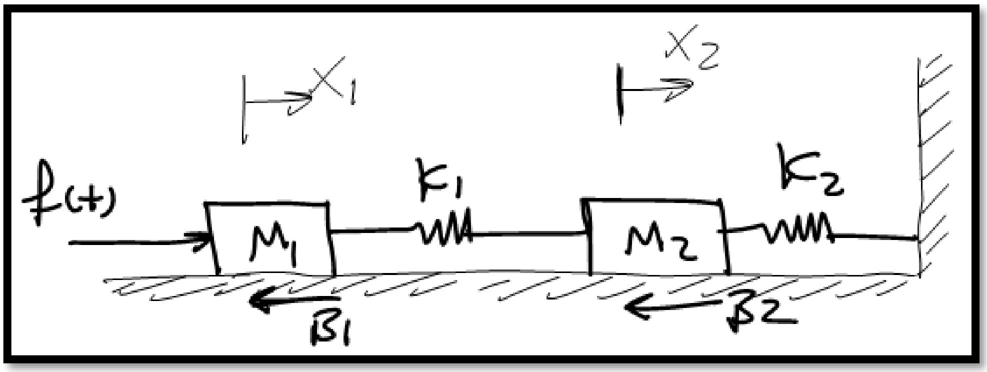
$$m\ddot{y} + \ b\dot{y} + \ ky = r$$

$$(ms^2 + bs + k)Y = r$$

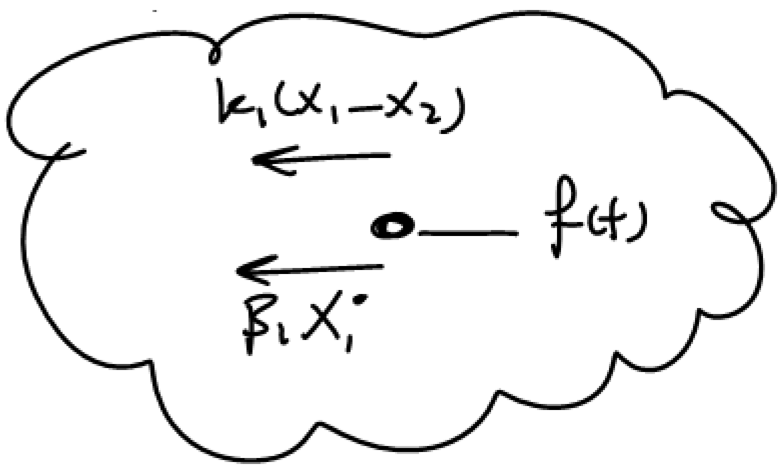
$$\frac{Y}{r} = \frac{1}{ms^2 + bs + k}$$

Changing system parameters (m, b, k) change the value and shape of the output signal

Example:



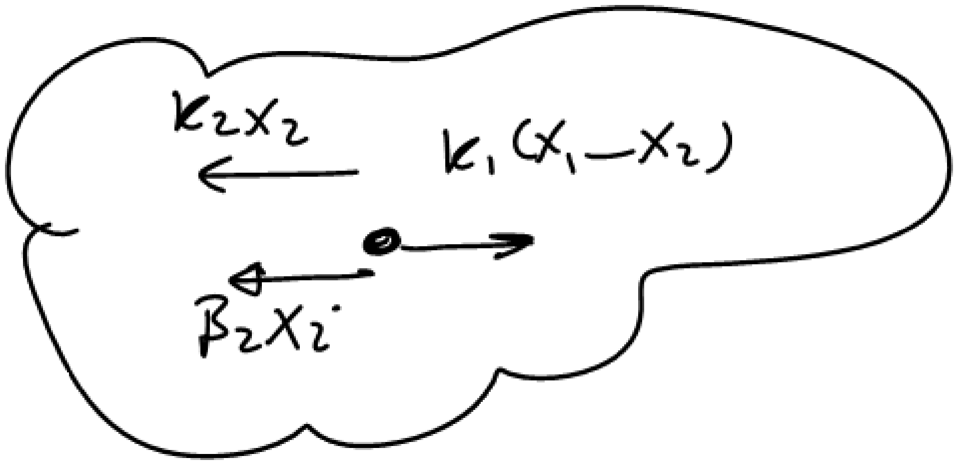
Free body diagram of mass M_1



$$M_1\ddot{x_1} = f - \beta_1\dot{x_1} - k_1(x_1 - x_2)$$

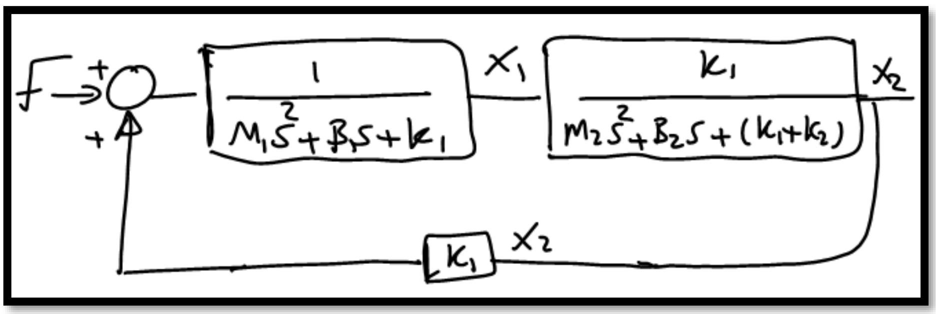
$$\therefore \quad F = (M_1 s^2 + \beta_1 s + k_1)X_1 - k_1 X_2$$

Free body diagram of mass M_2



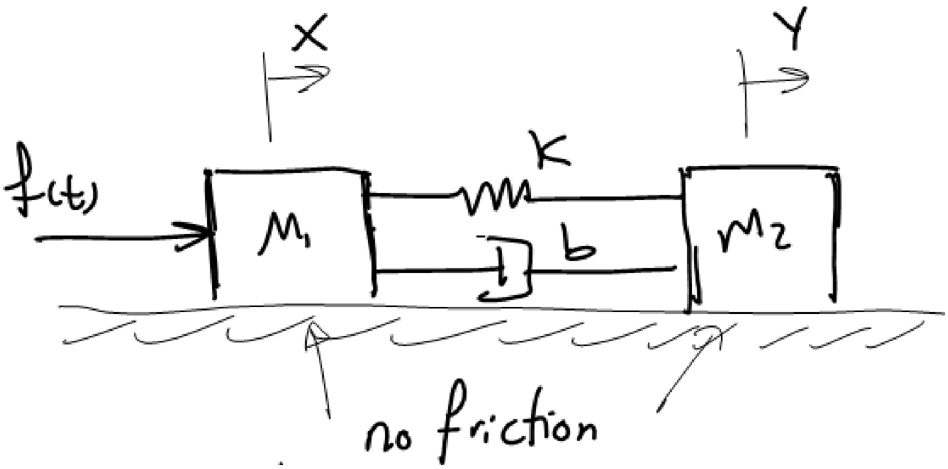
$$M_2\ddot{x_2} = k_1(x_1 - x_2) - \beta_2\dot{x_2} - k_2x_2$$

$$\therefore \quad k_1X_1 = (M_2s^2 + \beta_2 s + k_1 + k_2)X_2$$



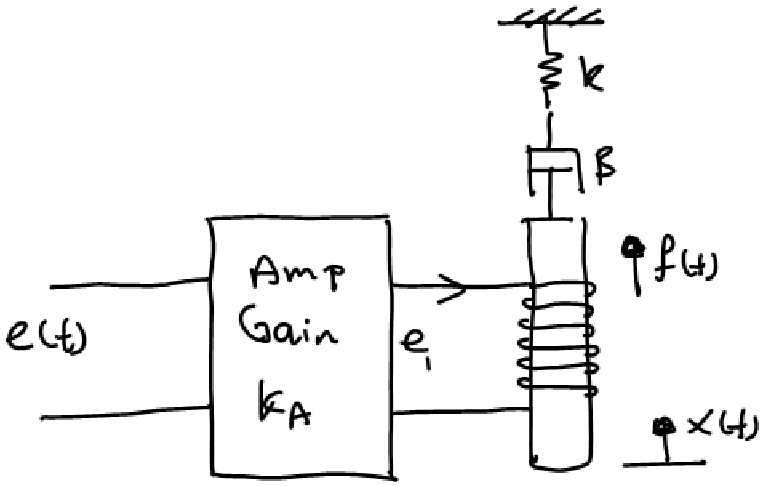
Problem:

Sketch the block diagram for the mechanical system shown below



Example:

Find the transfer function $\frac{X(s)}{E(s)}$

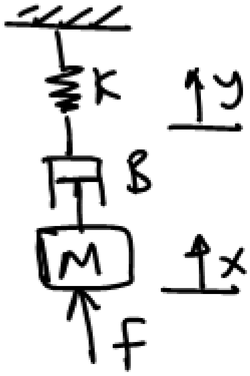


$E_1 = k_A E$

$E_1 = I(R + sL)$

$F = (ms^2 + \beta s)X - \beta sY$

$\beta sY = (\beta s + k)Y$



Please continue

Example:

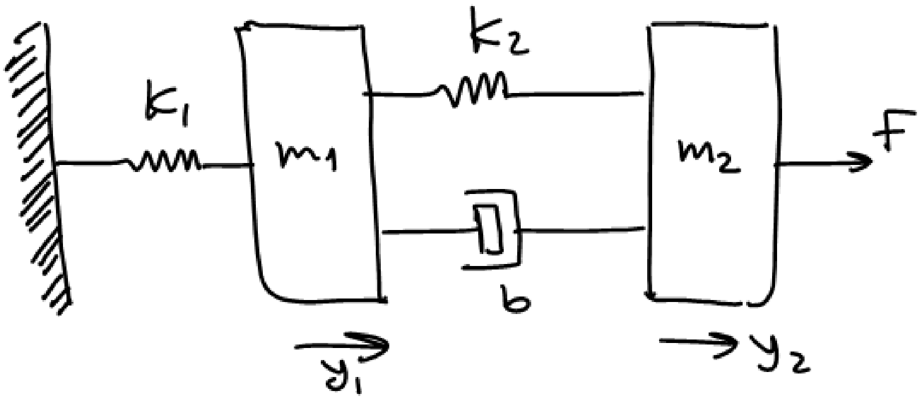
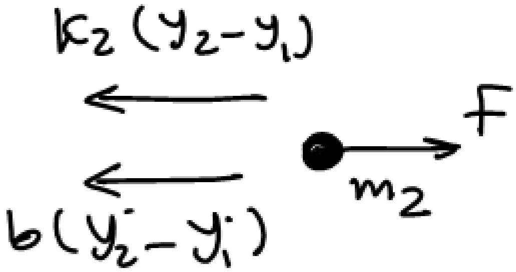
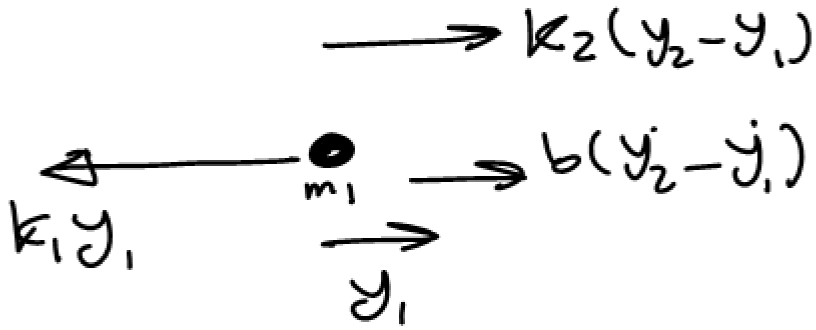


Figure above shows a mechanical system with two masses and two springs. Obtain the mathematical model of the system



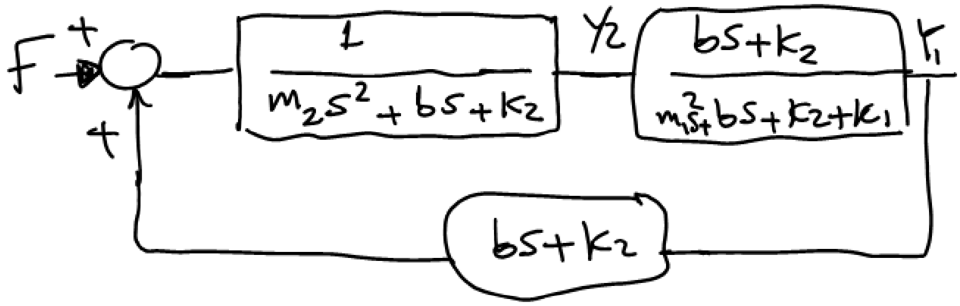
$$f - b(\dot{y}_2 - \dot{y}_1) - k_2(y_2 - y_1) = m_2\ddot{y}_2$$

$$F = (m_2 s^2 + bs + k_2)Y_2 - (bs + k_2)Y_1$$

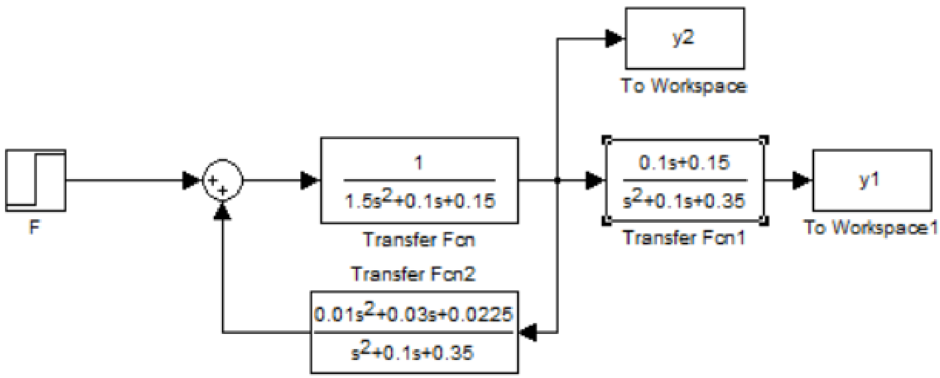


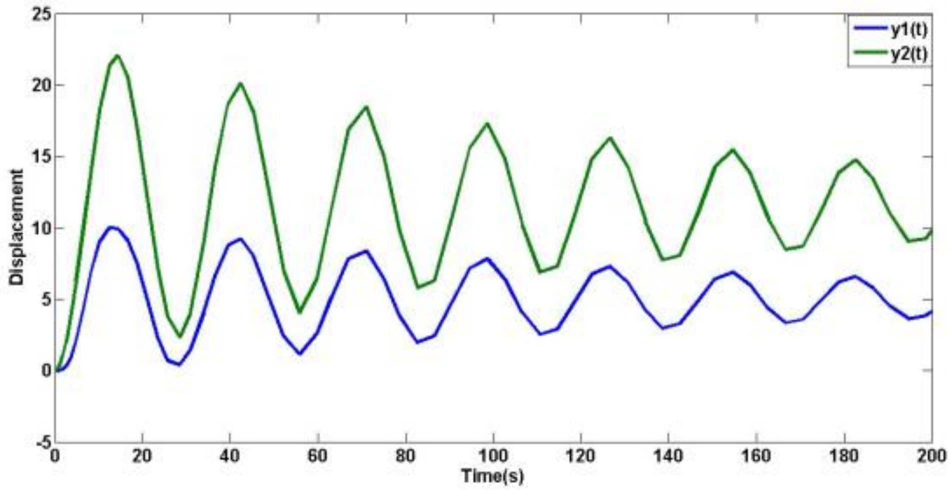
$$b(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - y_1) - k_1 y_1 = m_1 \ddot{y}_1$$

$$(bs + k_2)Y_2 = (m_1 s^2 + bs + k_2 + k_1)Y_1$$



Assume $f = 1$, $m_1 = 1$, $m_2 = 1.5$, $b = 0.1$, $k_1 = 0.2$, $k_2 = 0.15$





■ Modeling in electrical circuit

Electric circuits are frequently used in control systems largely because of the ease of manipulation and processing of electric signals

Although controllers are increasingly implemented with digital logic, many functions are still performed with analog circuit

1. Analog circuits are fasters than digital, and for very simple controllers, an analog circuit would be less expensive than a digital implementation
2. Furthermore, the power amplifier for electromechanical control for digital control must be analog circuit

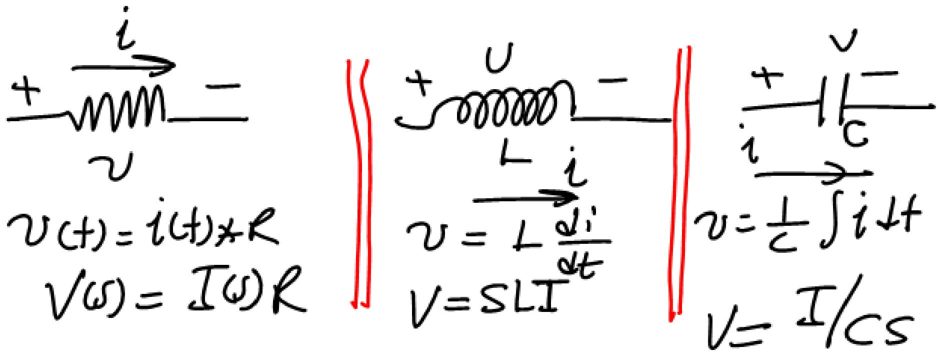
- **Kirchhoff's laws**

Kirchhoff current law: the algebraic sum of currents leaving a junction or node equals the algebraic sum of currents entering that node

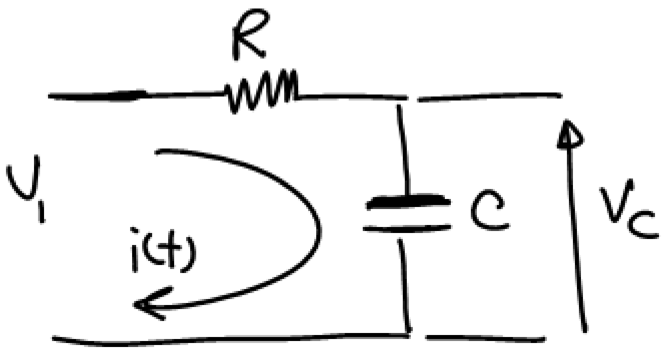
$$\boldsymbol{I_{in} = I_{out}}$$

Kirchhoff voltage law: the algebraic sum of all voltages taken around a closed path in a circuit is zero

$$\sum \boldsymbol{voltage\ sources} = \sum \boldsymbol{voltage\ drops}$$



Example:



$$v_1(t) = i(t)\,R + \frac{1}{C}\int i(t)\,dt$$

Taking Laplace

$$V_1(s) = I(s)\,R + \frac{I(s)}{Cs}$$

$$V_1(s) = I(s)\left(R + \frac{1}{Cs}\right)$$

$$v_c(t) = \frac{1}{C}\int i(t)\,dt$$

$$V_c(s) = \frac{I(s)}{Cs}$$

$$\therefore \quad \frac{V_c}{V_1} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1 + RCs}$$

Mathematically ➔ analytically

Assume $v_1(t) = u(t) = 1\ volt$

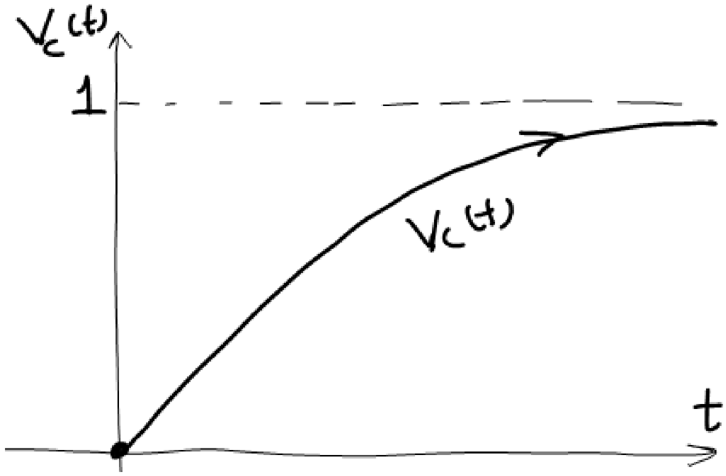
$$V_1 = 1/s$$

$$V_c = \frac{1}{s} - \frac{1}{s + 1/RC}$$

$$v_c(t) = 1 - e^{-t/RC}$$

Assume $R = 1,\ C = 0.0001$

$v_c(t) = 1 - e^{-10000t}$



charging capacitor

② by MATLAB ⇒ .m file

$$\frac{V_c}{V_1} = \frac{1}{1 + RCs}$$

Assume

⇓

$V_1 = u(t)$ ⇒ unit step

$R = 1\ \Omega$

$C = 0.0001$

```
R=1;
C=0.0001;
num = 1;
den = [R*C   1];
sys = tf(num, den);
t = 0 : 0.001 : 0.01;
Vc = step(sys, t);
plot(t, Vc)
```

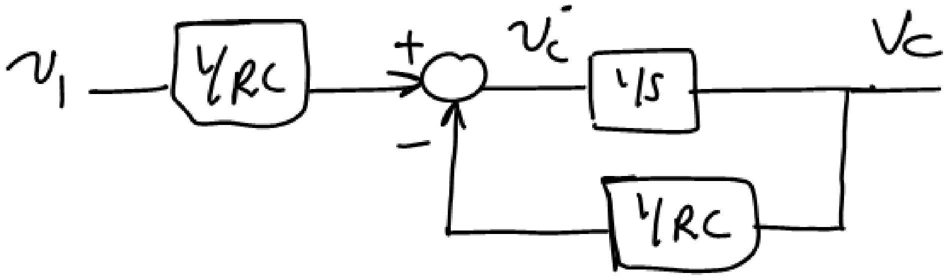
By SIMULINK

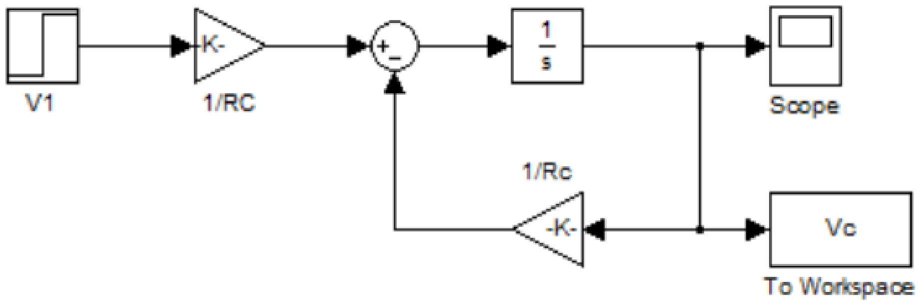
$$\frac{V_c}{V_1} = \frac{1}{1 + RCs}$$

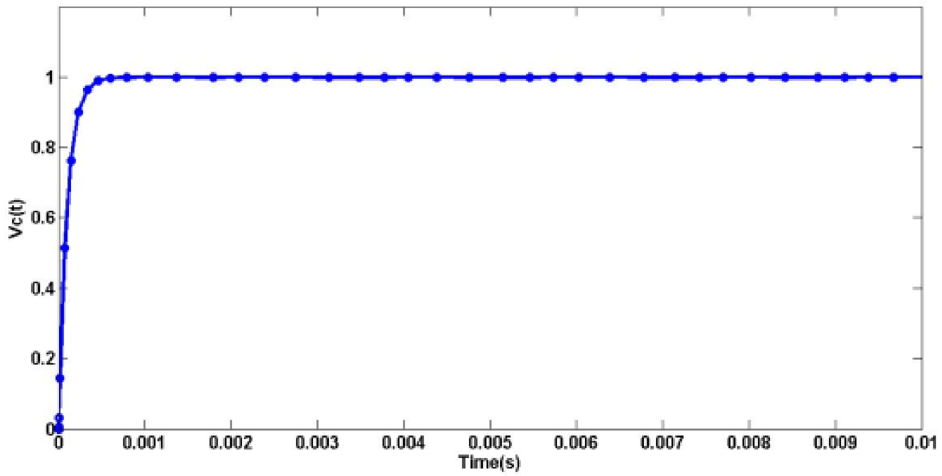
Taking inverse Laplace

$$v_c + RC\dot{v}_c = v_1$$

$$\frac{1}{RC}v_1 + \frac{1}{RC}v_c = \dot{v}_c$$

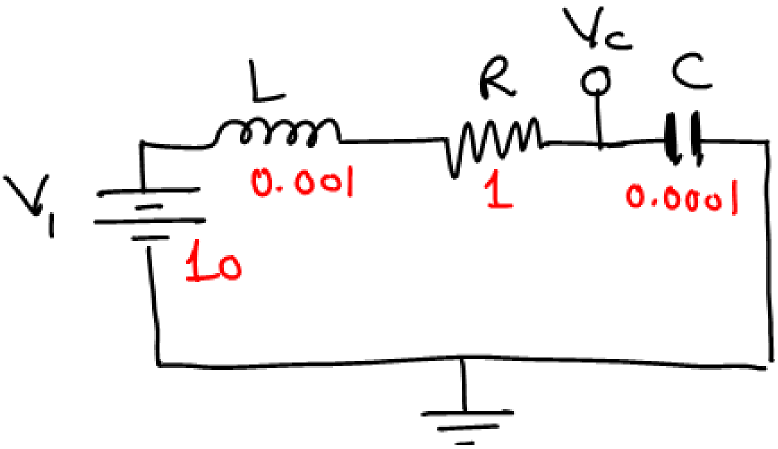






Example:

Use the SIMULINK to solve the following RLC circuit



$$v_1(t) = L\frac{di(t)}{dt} + i(t)\,R + \frac{1}{C}\int i(t)\,dt$$

Take Laplace

$$V_1(s) = LsI(s) + \;I(s)\,R + \frac{I(s)}{Cs}$$

$$V_1 = I\left(R + sL + \frac{1}{Cs}\right) = \frac{LCs^2 + RCs + 1}{Cs}$$

$$v_c(t) = \frac{1}{C}\int i(t)\,dt$$

$$V_c(s) = \frac{I(s)}{Cs}$$

$$\therefore \quad \frac{V_c}{V_1} = \frac{1}{LCs^2 + RCs + 1}$$

$$\frac{V_c}{V_1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

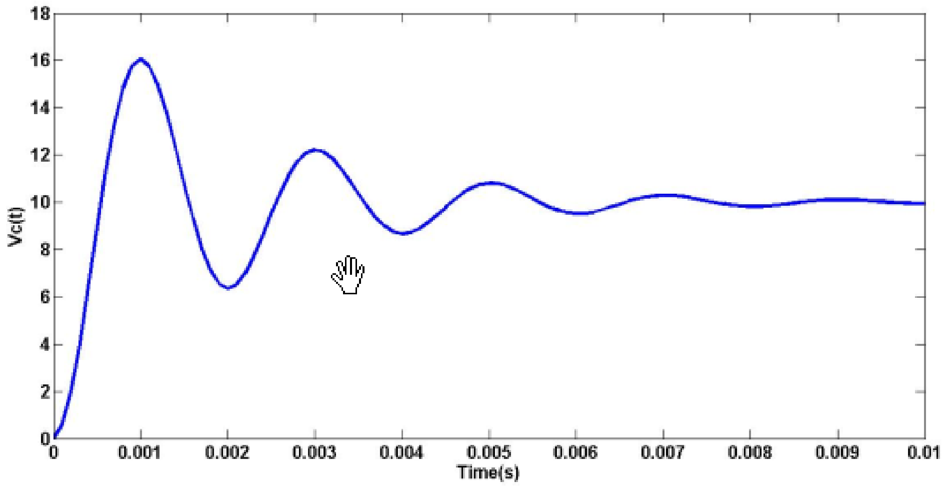
Assume $v_1(t) = 10\,u(t)$, $R = 1$, $C = 0.0001$, $L = 0.001$

Analytically

$$V_c = \frac{10^8}{s^2 + 1000s + 10^8}$$

$$= \frac{10}{s} - 10\frac{s + 500}{(s + 500)^2 + 9750000} + \frac{4000}{(s + 500)^2 + 9750000}$$

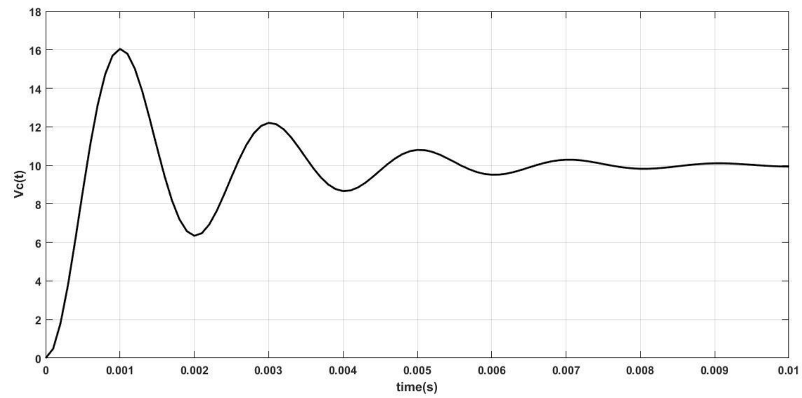
$$v_c(t) = 10 - 10e^{-500t}\cos\sqrt{9750000}t - \frac{4000}{\sqrt{9750000}}e^{-500t}\sin\sqrt{9750000}t$$



BY MATLAB // .m file

$$\frac{V_c}{V_1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

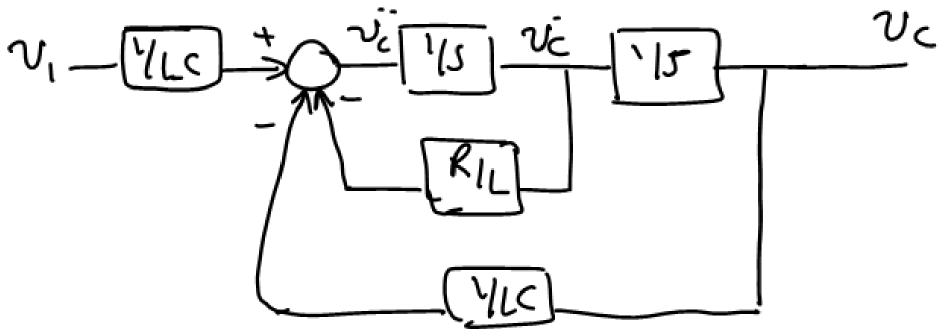
```
R =1;
L = 0.001;
C = 0.0001;
num = 1/(L*C);
den = [1 R/L 1/(L*C)];
sys = tf(num,den);
t = 0:0.0001:0.01;
Vc = step(10*sys,t);
plot(t,Vc)
```

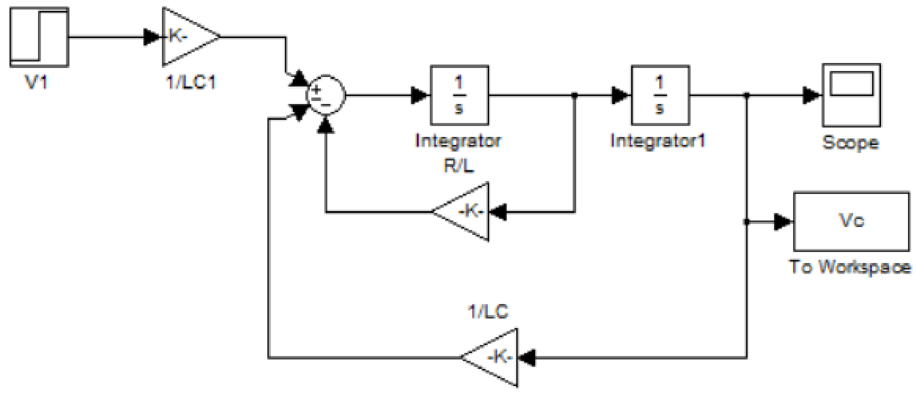


BY SIMULINK

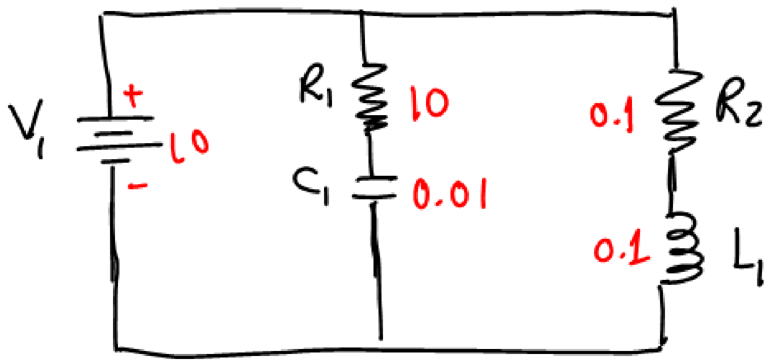
$$\frac{V_c}{V_1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\ddot{v}_c + \frac{R}{L}\dot{v}_c + \frac{1}{LC}v_c = \frac{1}{LC}v_1$$

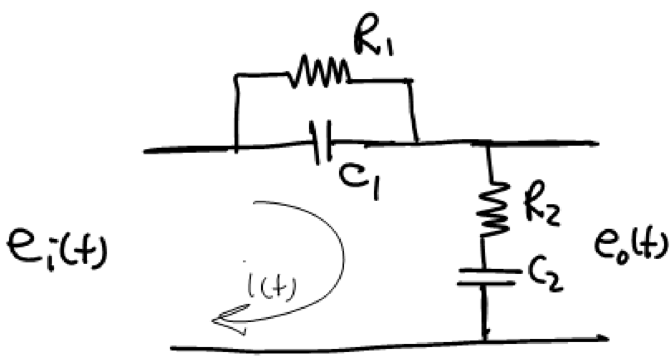




Problem:

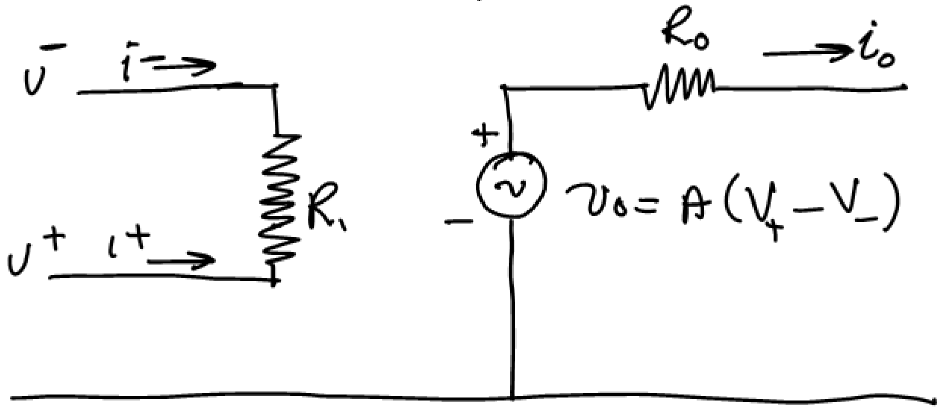


Problem:

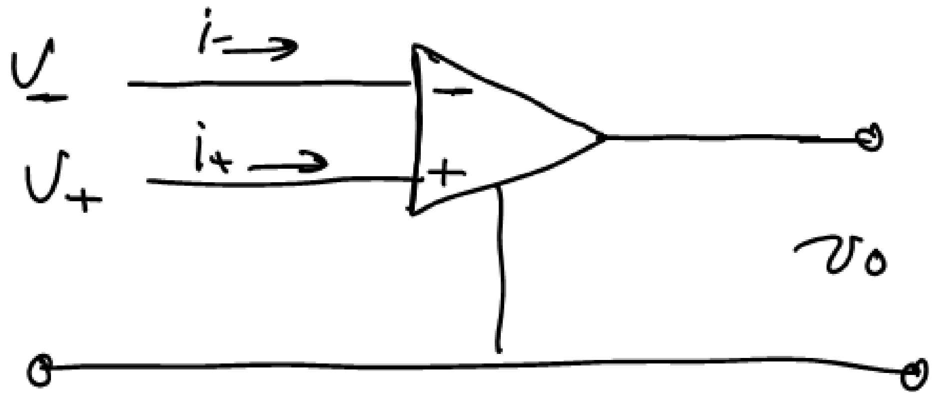


- Operational Amplifier

The simplified circuit of the Op-Amp is shown

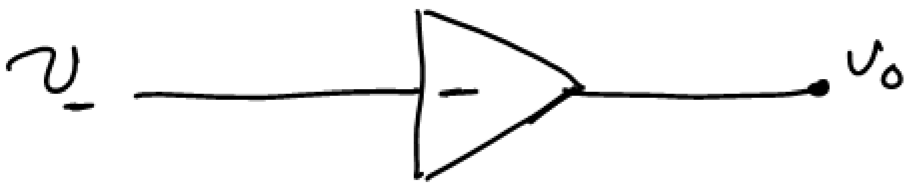


Schematic symbol



If the positive terminal is not shown, it is assumed to be connected to ground, $v^+ = 0$

And the reduced symbol below is used



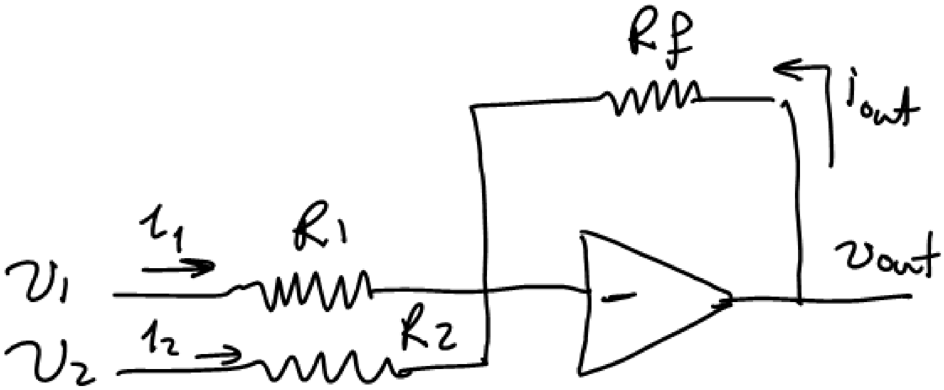
It is usually assumed that the OP-Amp is ideal with the values $R_{in} = \infty, R_{out} = 0, A = \infty$

$i^+ = i^- = 0$

$v^+ = v^-$

Example:

Find the equation and the transfer function of the circuit shown



$\because \quad v^+ = 0$

$\therefore \quad v^- = 0$

$$i_1 = \frac{v_1}{R_1}$$

$$i_2 = \frac{v_2}{R_2}$$

$$i_{out} = \frac{v_{out}}{R_f}$$

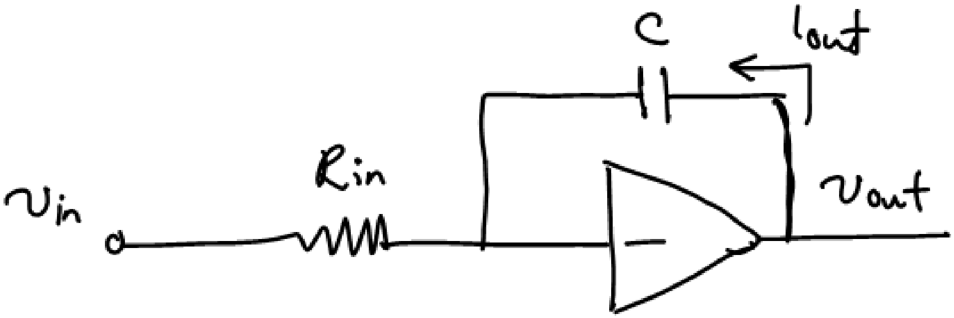
$i_1 + i_2 + i_3 = 0$

$$\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_{out}}{R_f} = 0$$

$$\therefore \quad v_{out} = -\left[\frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2\right]$$

The circuit O/P is a weighted sum of the input voltages with a sign change ➔ **summer circuit**

Example: Integrator



$\because \quad v^+ = 0$

$\therefore \quad v^- = 0$

$$\frac{v_{in}}{R_{in}} + C\frac{dv_{out}}{dt} = 0$$

$$\therefore \quad \frac{dv_{out}}{dt} = -\frac{v_{in}}{R_{in}C}$$

$$\frac{dv_{out}}{dt} = -\frac{1}{R_{in}C}\int_0^t v_{in}(\tau)\,d\tau + v_{out}(0)$$

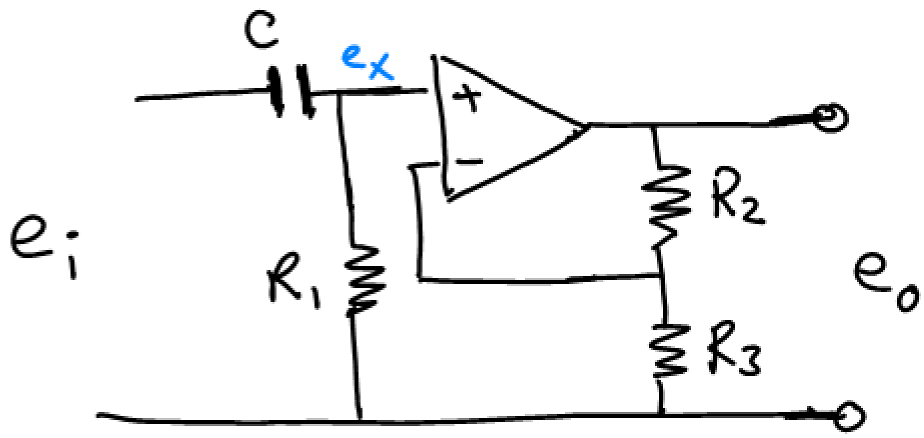
OR

$$sv_{out} = -\frac{1}{R_{in}C}v_{in}$$

$$\therefore \quad \frac{v_{out}}{v_{in}} = -\frac{1}{R_{in}C}$$

Example:

Find E_0/E_i



$$\frac{E_x}{E_i} = \frac{R_1}{R_1 + 1/sC}$$

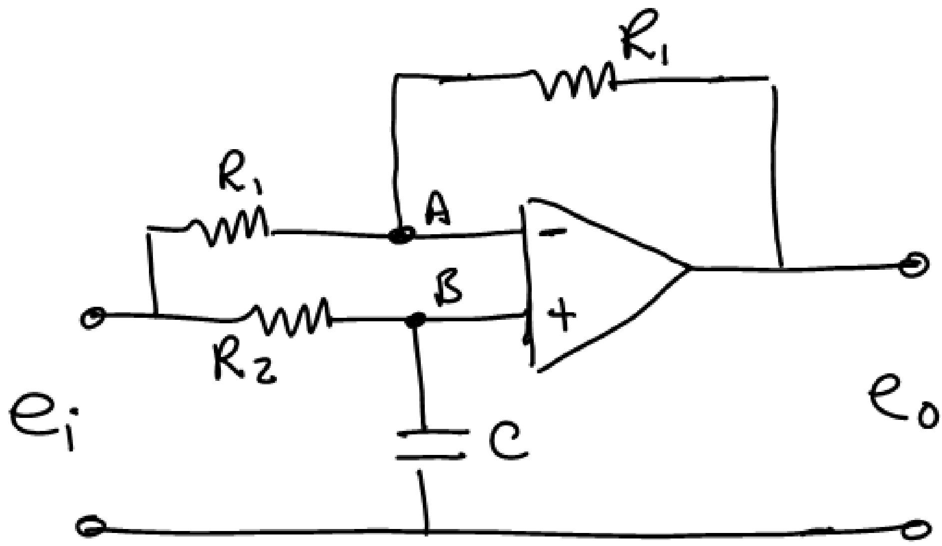
$$\frac{E_x}{E_o} = \frac{R_3}{R_2 + R_3}$$

$$\frac{E_o}{E_i} = \frac{R_1}{R_1 + 1/sC} \, \frac{R_2 + R_3}{R_3}$$

$$\frac{E_o}{E_i} = \frac{R_1(R_2 + R_3)}{R_3(R_1 + 1/sC)}$$

Problem:

Find E_0/E_i



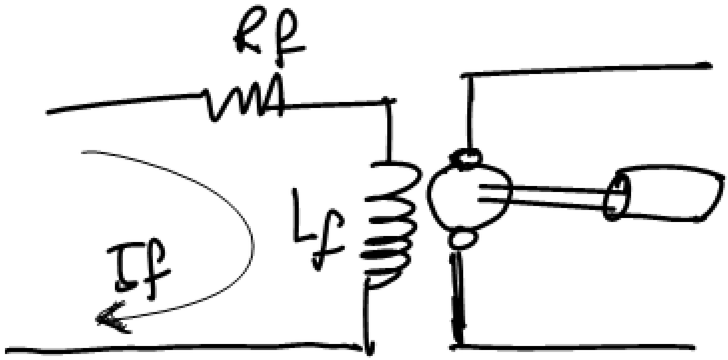
■ DC Motor

$$Torque \propto i_a i_f$$

Field control

i_a : constant

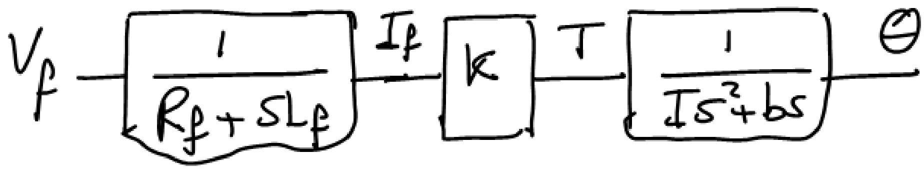
$Torque \propto i_f$



$V_f = I_f\left(R_f + sL_f\right)$

$T = kI_f$

$T = (Is^2 + bs)\theta$

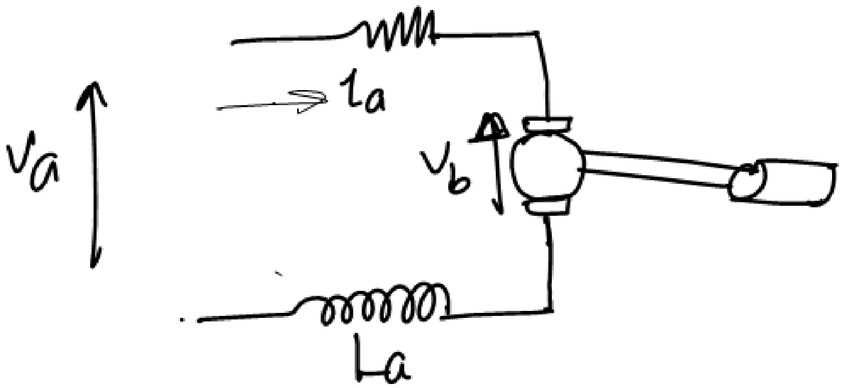


$$\frac{\theta}{V_f} = \frac{k}{\left(R_f + sL_f\right)(Is^2 + bs)}$$

Armature control

i_f : constant

$Torque \ \propto i_a$



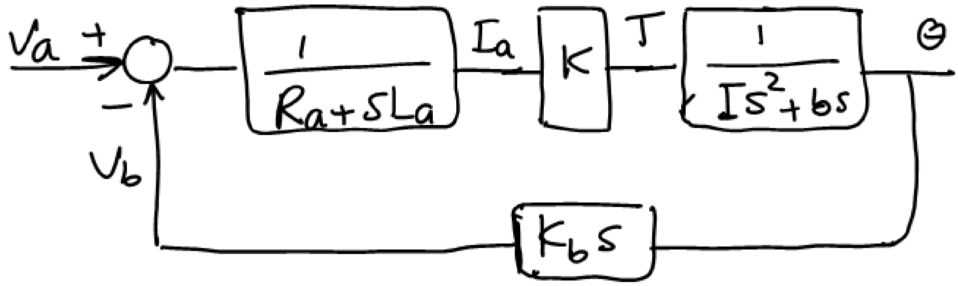
$V_a - V_b = I_a(R + sL_a)$

$v_b \propto \dot{\theta}$

$V_b = k_b s\,\theta$

$T = kI_a$

$T = \ (Is^2 + bs)\theta$



Chapter 05

Block Diagram

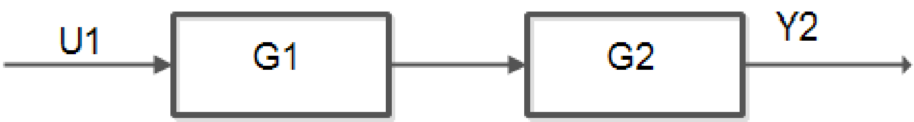
Table of content

- Block Diagram
- MATLAB session
- SIMULINK

■ Block Diagram

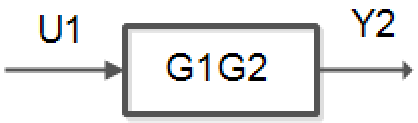
- In many control systems, the system equations can be written so that their components do not interact except by having the input of a part be the output of another part
- In these cases, it is easy to draw a block diagram that represents the mathematical relationships in a manner similar to that used for the component block diagram
- The transfer function of each component is placed in a box
- And the I/O relationships between components are indicated by lines and arrows
- We can then solve the equations by graphical simplification which is often easier and more informative than algebraic manipulation

Series

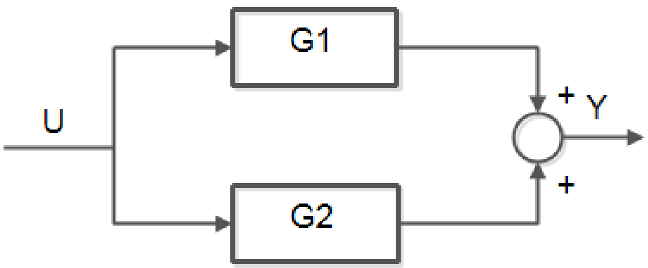


$$U1 * G1 * G2 = Y2$$

$$\frac{Y2}{U1} = G1 * G2$$



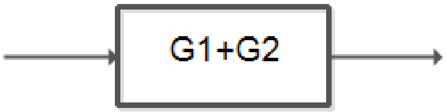
Parallel:



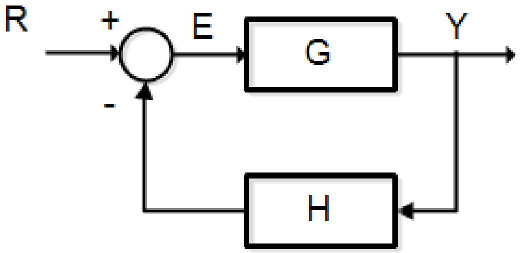
$$U * G1 + U * G2 = Y$$

$$U(G1 + G2) = Y$$

$$\frac{Y}{U} = G1 + G2$$



Feedback:



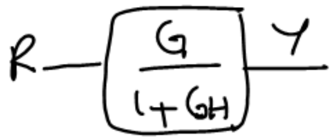
$$E = R - YH$$

$$EG = Y$$

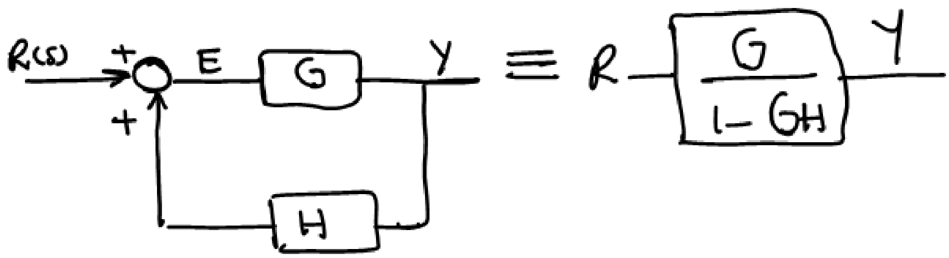
$$(R - YH)G = Y$$

$$RG = Y(1 + GH)$$

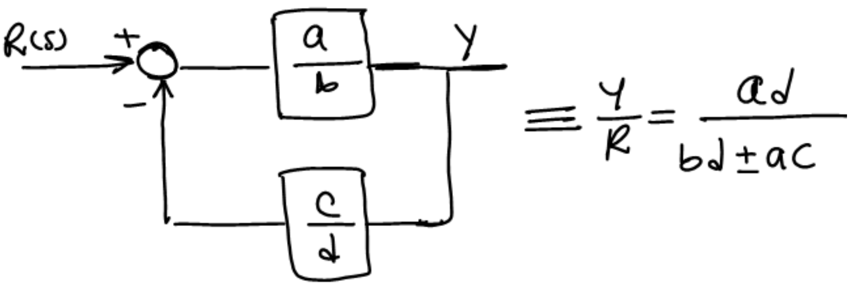
$$\frac{Y}{R} = \frac{G}{1 + GH}$$



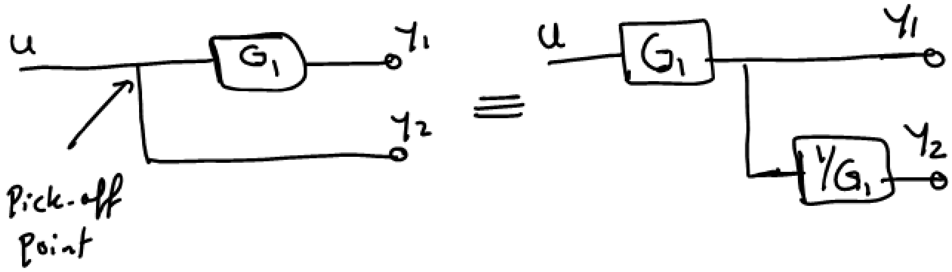
Similarly



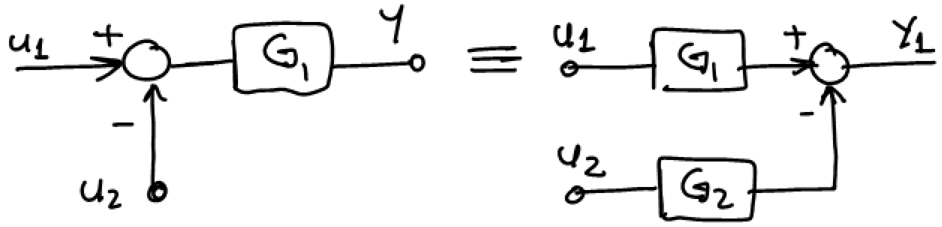
Hint:

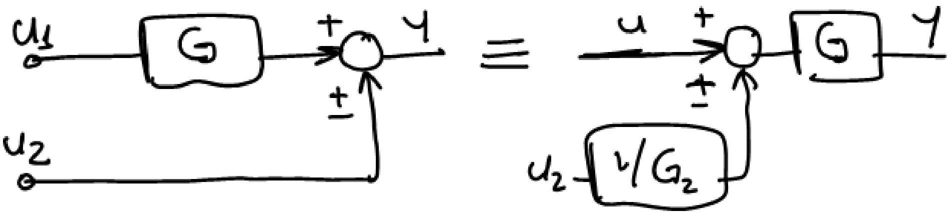


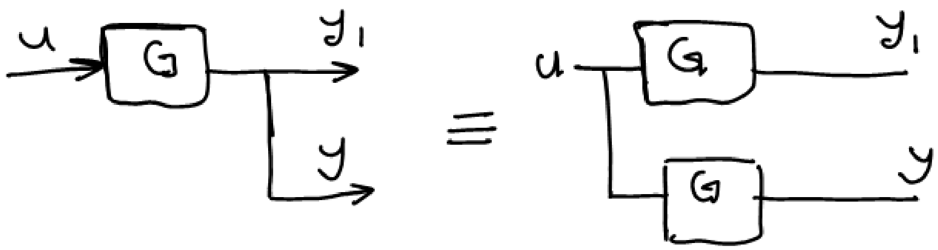
Note that:

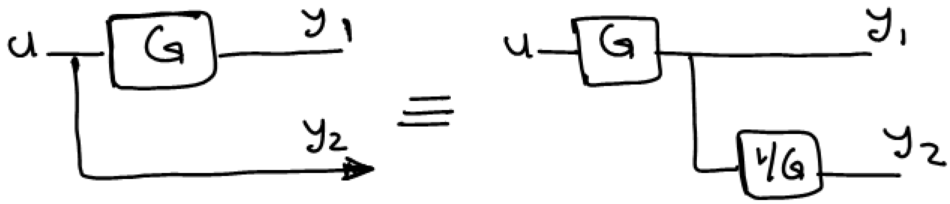


You can think of it as taking $G1$ as common factor or you want to move the pick-off point to the right



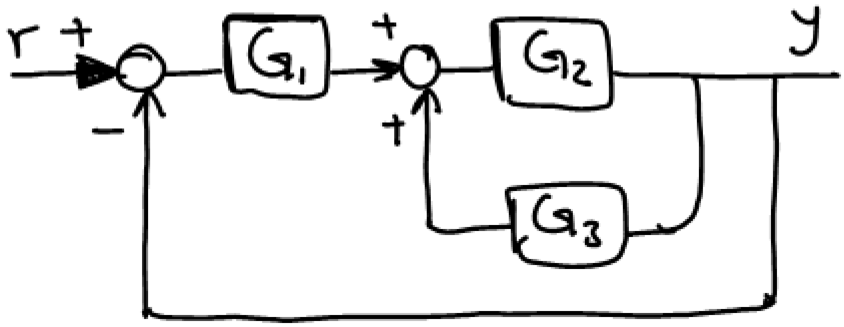


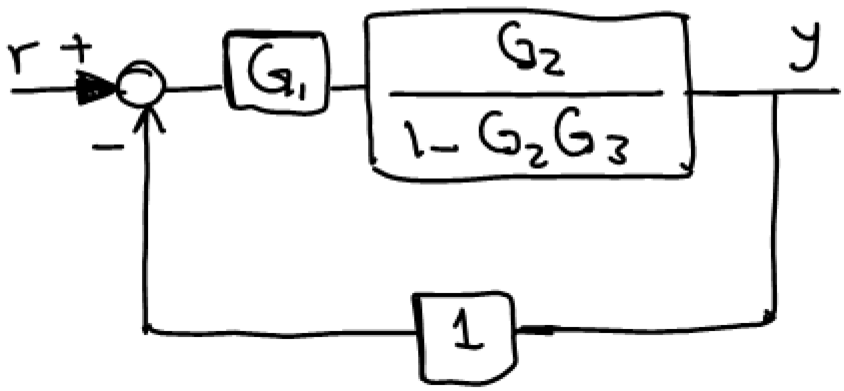


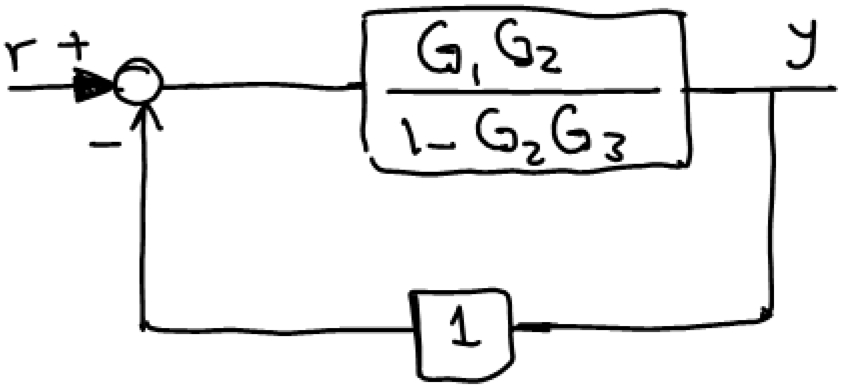


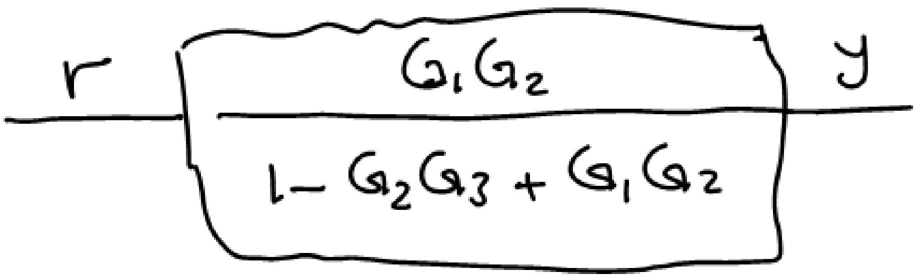
Example:

Find the transfer function Y/R



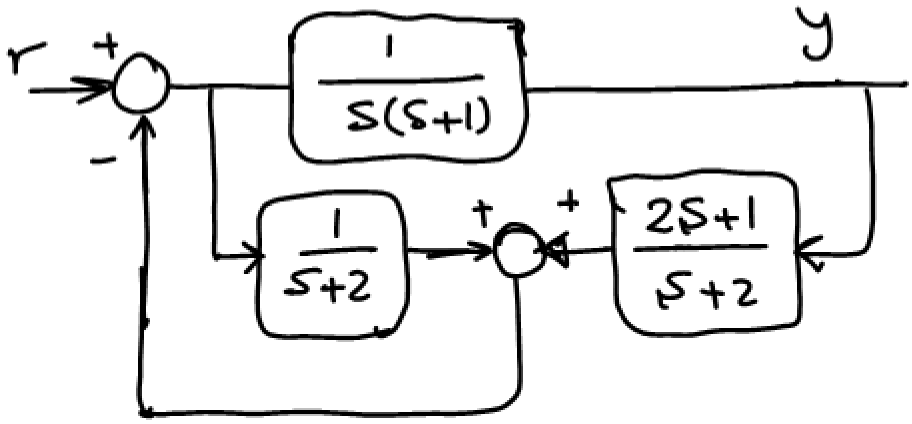


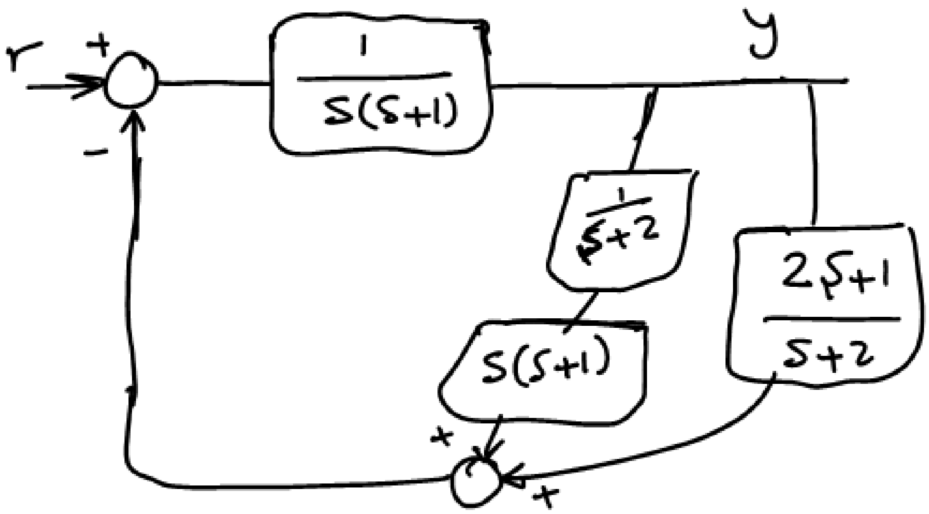


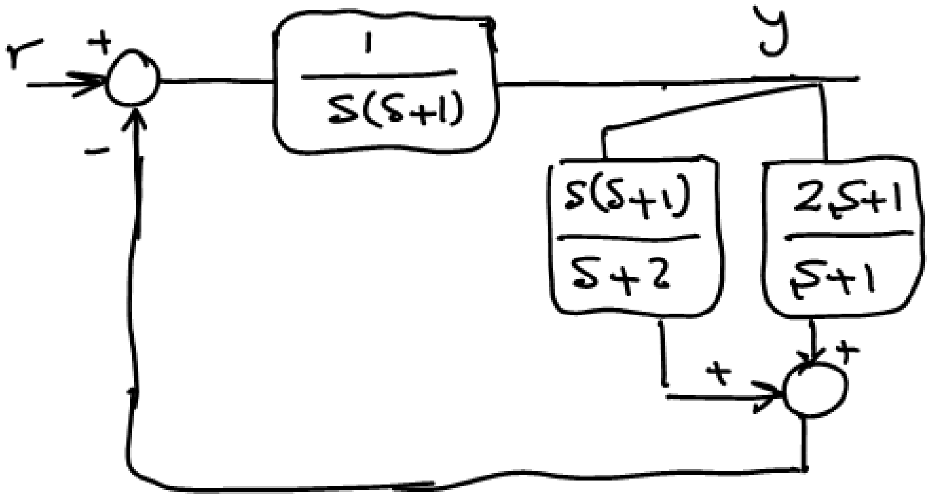


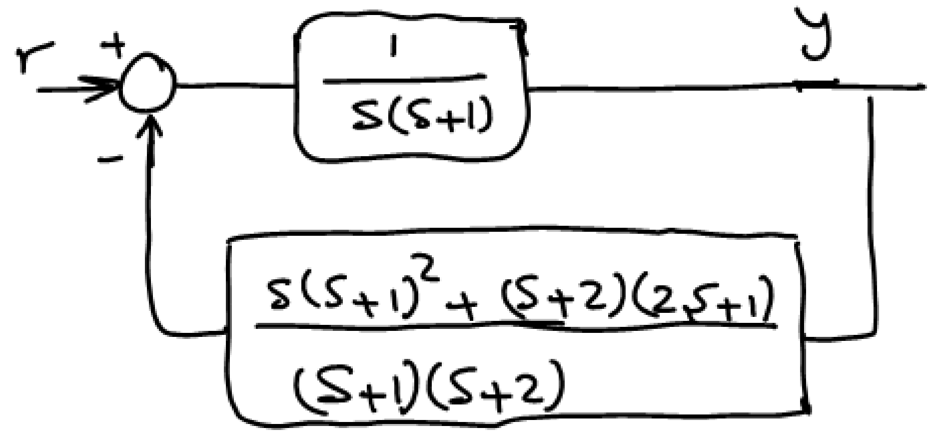
Example:

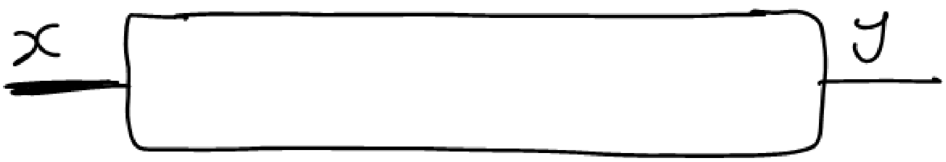
Find the transfer function Y/R



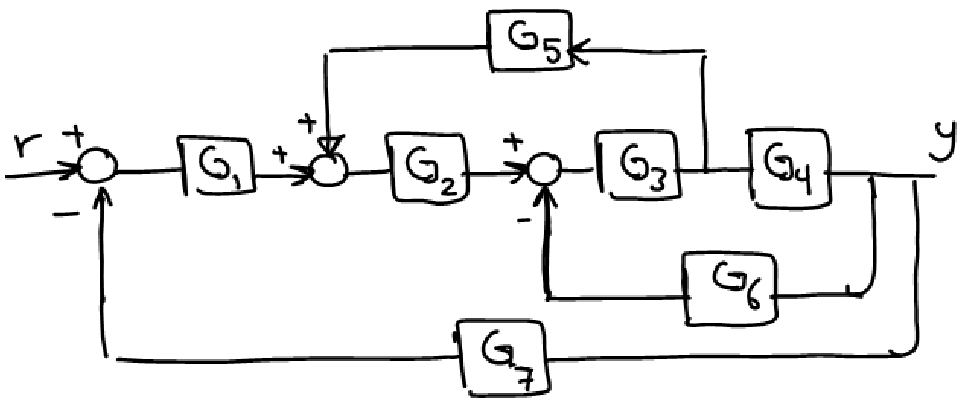

r
+
−
1/s(s+1)
y
1/s+2
s(s+1)
2s+1/s+2
+
+


r
+
−
1/s(s+1)
y
s(s+1)/s+2
2s+1/s+1
+
+



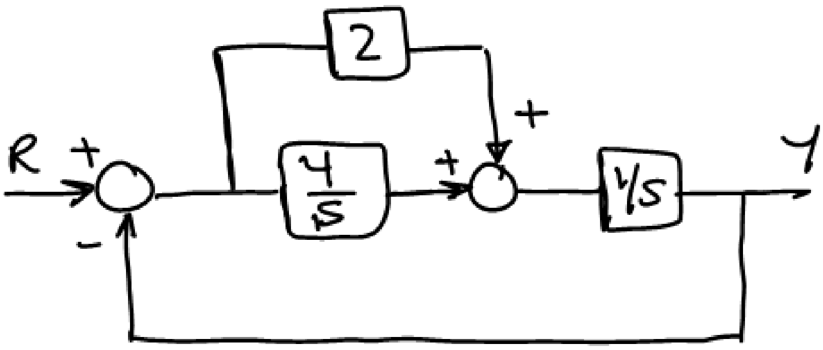


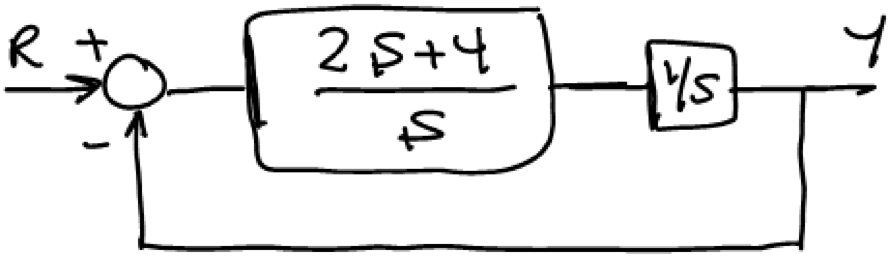
Problem:

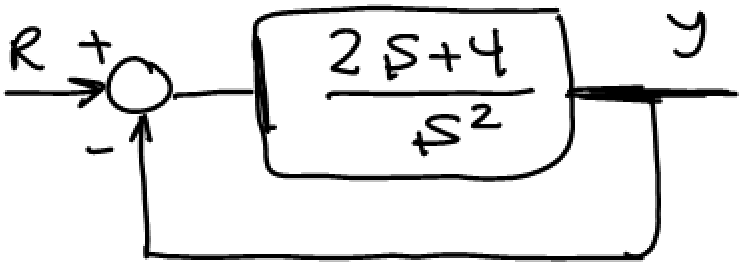


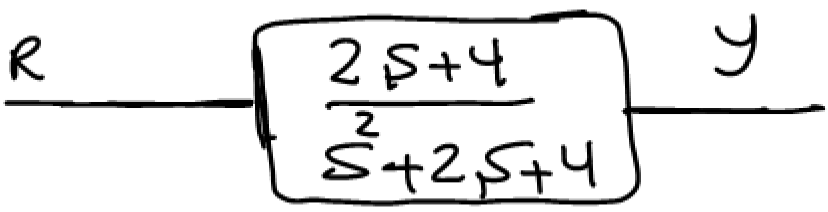
Example:

Find the transfer function of the system below





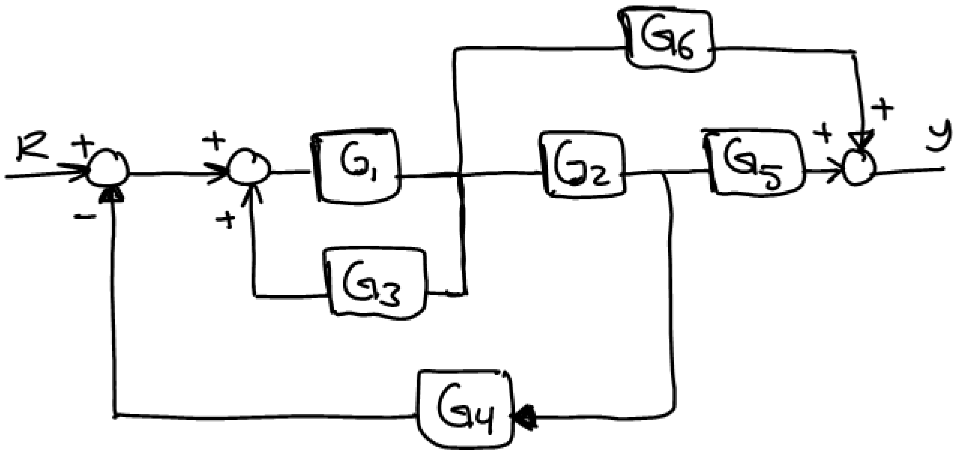


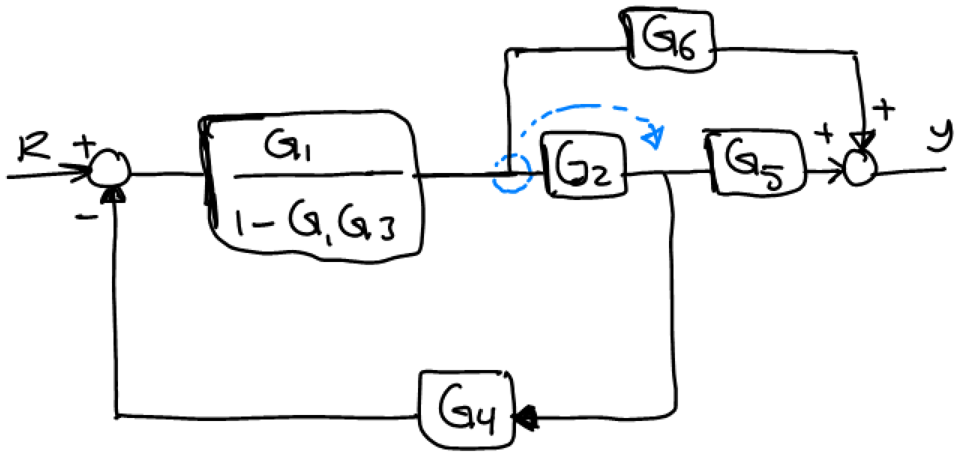


$$\frac{Y}{R} = \frac{2s+4}{s^2+2s+4}$$

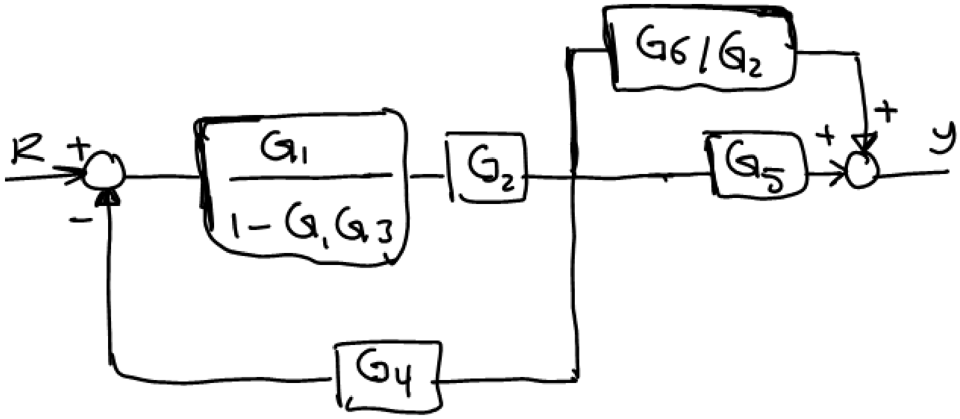
Example:

Find the transfer function of the system below





Move the pick off point preceding $G2$ to its output



Please continue to get

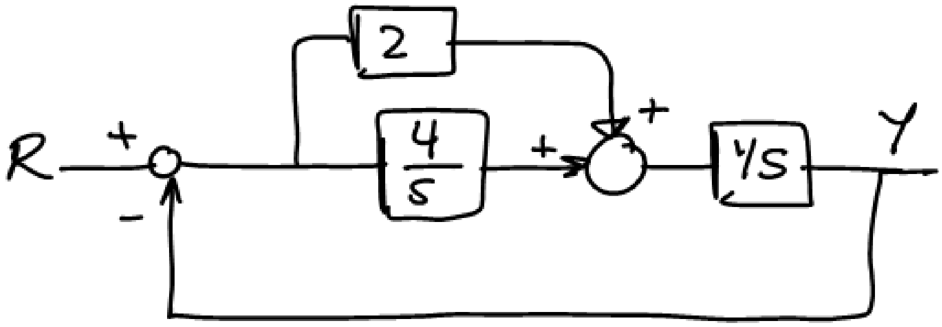
$$TF = \frac{G1\ G2\ G5 + G1\ G6}{1 - G1\ G3 + G1\ G2\ G4}$$

Block Diagram reduction using MATLAB

- Series
- Parallel
- Feedback

Example:

Find the transfer function of the system below using MATLAB




```
num = [2];
den = [1];
SysG1 = tf(num,den);
num = [4];
den = [1 0];
SysG2 = tf(num,den);
SysG3=parallel(SysG1,SysG2)
num = [1];
den = [1 0];
SysG4 = tf(num,den);
SysG5=series(SysG3,SysG4)
SysG6=feedback(SysG5,1)
```

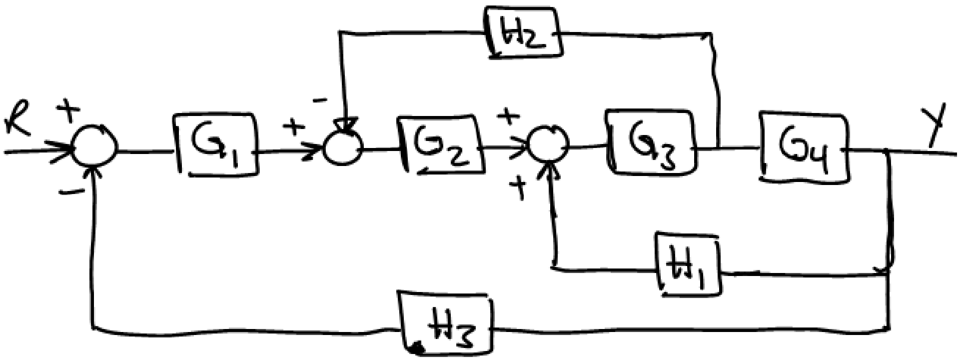
```
SysG6 =

     2 s + 4
  -------------
  s^2 + 2 s + 4

Continuous-time transfer function.
```

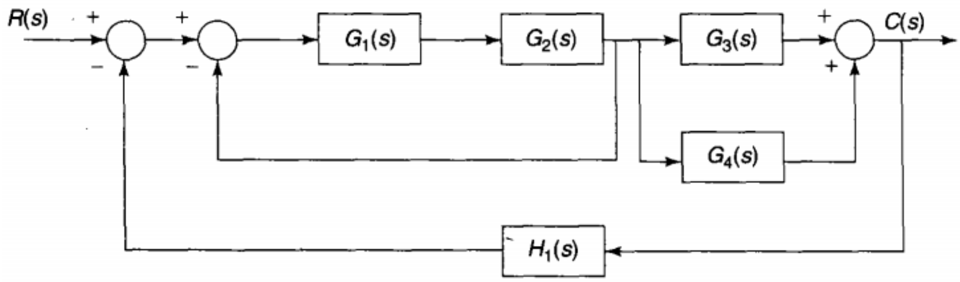
Problem:

Find the transfer function of the system below using MATLAB

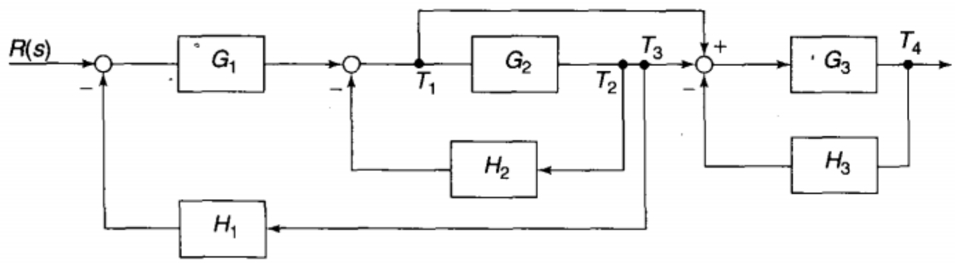


Problem:

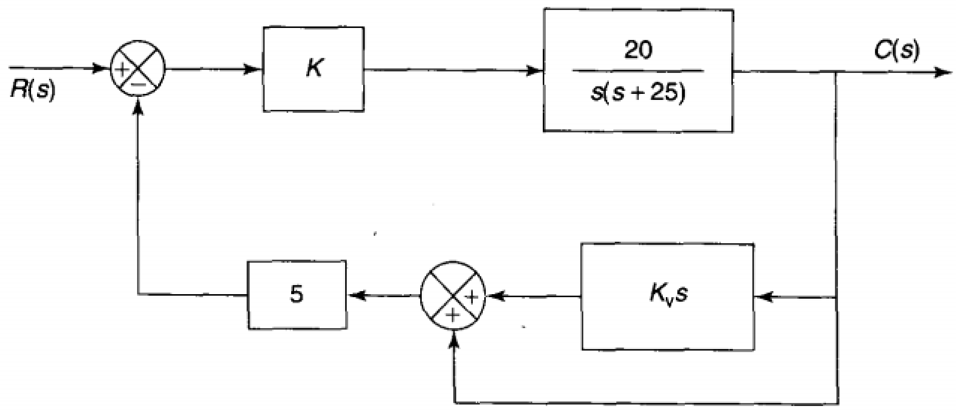
1. Reduce the system described below to a single block and determine the transfer function of that block



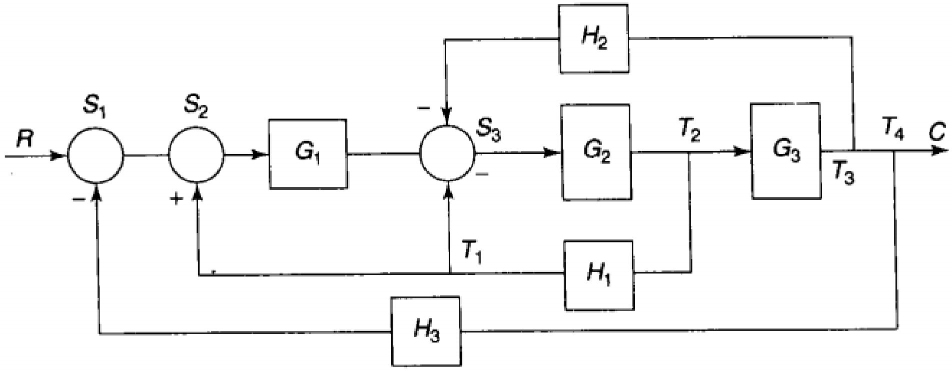
2. Find the transfer function



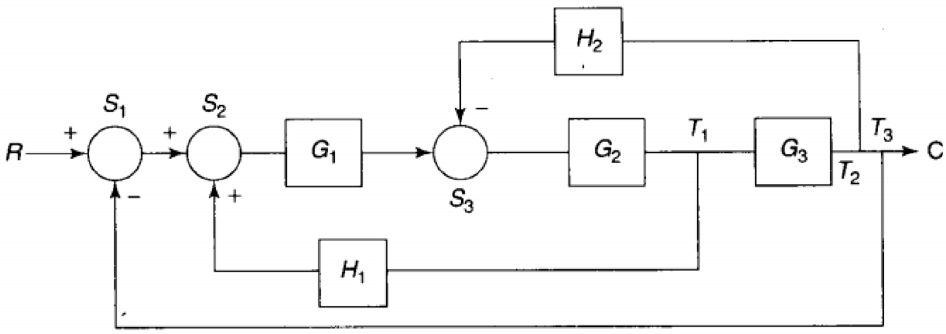
3. Find the transfer function



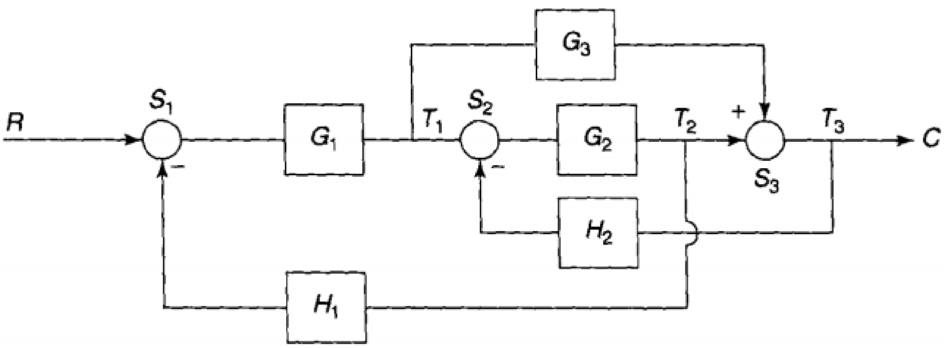
4. Simplify the block diagram for the system shown below and determine the closed-loop transfer function



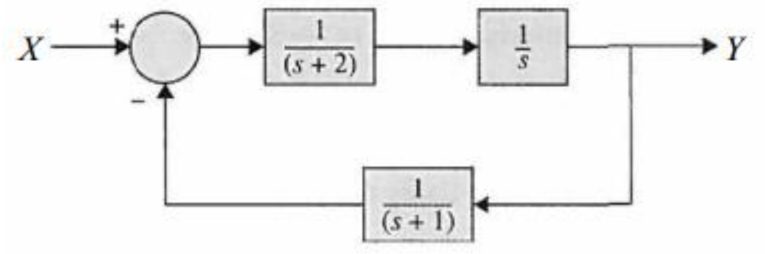
5. Obtain $C(s)/R(s)$



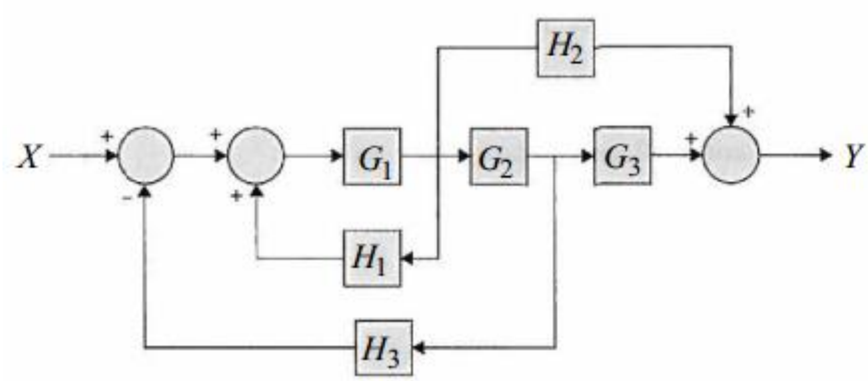
6. Reduce the block diagram shown below and obtain $C(s)/R(s)$



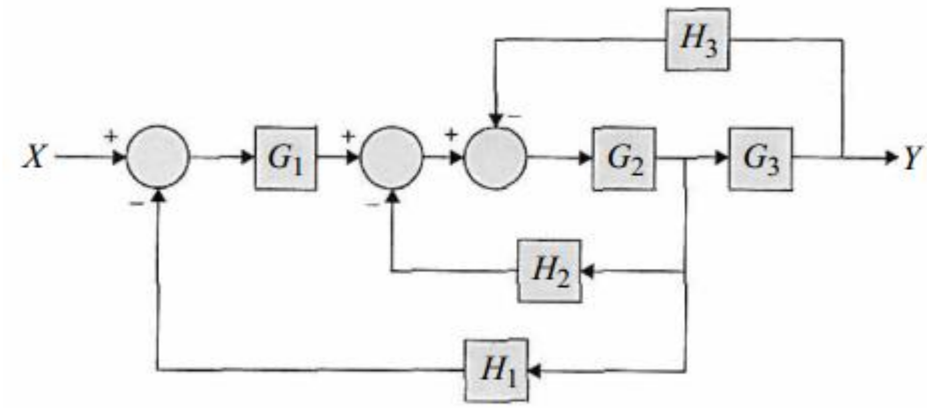
7. Reduce the block diagram shown below to unify feedback form and find the system **characteristic equation**



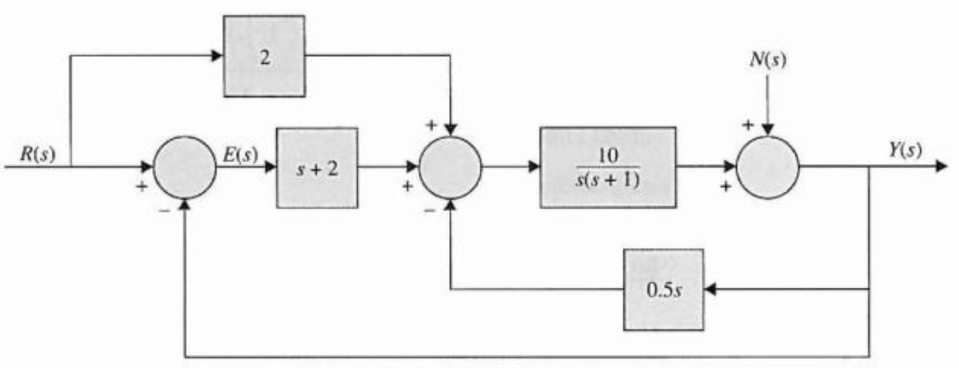
8. Reduce the block diagram shown and find Y/X



9. Reduce the block diagram shown and find Y/X



10. Find the output $Y(s)$ when $R(s)$ and $N(s)$ are applied simultaneously



Chapter 06

Time Response

Table of content

- Test signal
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  - Dynamic analysis – transient response
    - 1^{st}-order system subjected to step input
    - 1^{st}-order system subjected to ramp input
    - 2^{nd}-order system
  - Pole – zero – s-plane
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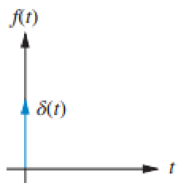
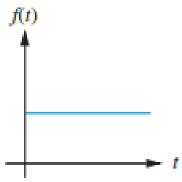
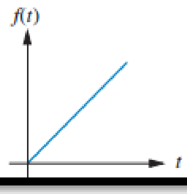
▪ Introduction

- Once mathematical model has been obtained for a control system, a number of techniques can be applied to analyze the system's performance.
- The next step following block diagram reduction, is **analysis and design**. If you are interested only in the performance of an individual subsystem, you can skip the block diagram reduction and move immediately into analysis and design. In this phase, the engineer analyzes the system to see if the **response specifications and performance requirements can be met by simple adjustments of system parameters**. If specifications cannot be met, the designer then designs additional hardware "**Controller**" in order to affect a desired performance.
- One of the important analysis techniques is the time-domain analysis
- The manner in which a **dynamic system responds to an input, expressed as a function of time**, is called the **time response**
- Static response characteristic, such as steady-state errors and transient response characteristic for various input signals are covered

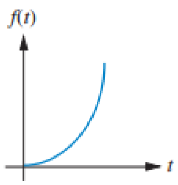
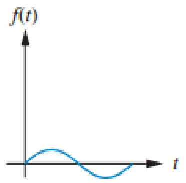
▪ Test signal

Test input signals are used, both analytically and during testing, to verify the design. It is neither necessarily practical nor illuminating to choose complicated input signals to analyze a system's performance. Thus, the **engineer usually selects standard test inputs**. These inputs are **impulses**, **steps**, **ramps**, **parabolas**, and **sinusoids**, as shown

Test waveforms used in control systems

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error

Which of these typical input signals to use for analyzing system characteristics may be **determined by the form of the input that the system will be subjected to most frequently under normal operation**. If the inputs to a control system are **gradually changing functions of time**, then a **ramp** function of time may be a good test signal. Similarly, if a system is subjected to **sudden disturbances**, a **step function** of time may be a good test signal; and for a system subjected to **shock inputs**, an **impulse** function may be best.

Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

An impulse is infinite at $t = 0$ and zero elsewhere. The area under the unit impulse is 1. An approximation of this type of waveform is used to place initial energy into a system so that the response due to that initial energy is only the **transient response of a system**. From this response the designer can derive a mathematical model of the system.

A **step input** represents a **constant command**, such as **position**, **velocity**, or **acceleration**. Typically, the step input command is of the **same form as the output**. For example, if the system's output is position, as it is for the antenna azimuth position control system, **the step input represents a desired position**, and the output represents the actual position. If the system's output is velocity, as is the spindle speed for a video disc player, the **step input represents a constant desired speed**, and the output represents the actual speed. The designer uses step inputs because both the transient response and the steady-state response are clearly visible and can be evaluated.

The **ramp input** represents a **linearly increasing command**. For example, if the system's output is position, the input ramp represents a linearly increasing position, such as that found when tracking a satellite moving across the sky at constant speed. If the system's output is velocity, the input ramp represents a linearly increasing velocity. **The response to an input ramp test signal yields additional information about the steady-state error**. The previous discussion can be extended to **parabolic inputs**, which are also used to evaluate a system's steady-state error.

Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory. The use of such test signals **enables one to compare the performance of many systems on the same basis**

The control systems engineer must take into consideration other characteristics about feedback control systems. For example, **control system behavior is altered by fluctuations in component values or system parameters**. These variations can be **caused by temperature, pressure**, or other environmental changes. Systems must be built so that **expected fluctuations do not degrade performance beyond specified bounds**. A **sensitivity analysis** can yield the percentage of change in a specification as a function of a change in a system parameter. One of the **designer's goals, then, is to build a system with minimum sensitivity over an expected range of environmental changes**

- the designer is concerned about **transient response**, **steady-state error**, **stability**, and **sensitivity**

■ Transient Response of a first-order system

Elements are said to be first order when the relationship between the input and the output **involves the rate at which the output changes**

$$\boldsymbol{a_1 \frac{dy}{dt} + a_0 y = b_0 x}$$

Where

$\boldsymbol{y}$: output ➔ measured quantity

$\boldsymbol{x}$: input quantity

$\boldsymbol{t}$: time

- We no longer have a simple relationship between the input and the output but a relationship which involves time
- the transducer that contains storage element cannot respond instantaneously to change an input.
    - The mercury in glass thermometer
    - Velocity of a true falling mass
    - Air pressure build-up in bellow
- To illustrate this, consider a **thermometer at temperature $\boldsymbol{T_0}$ inserted into a liquid at a temperature $\boldsymbol{T_1}$**. We can thus think of the thermometer being subject to step input (abruptly changes from **$\boldsymbol{T_0}$ to $\boldsymbol{T_1}$**). Thus, we have a measurement system, the thermometer, which has a step input and an output which changes from T_0 to T_1 over some time.

The question we are concerned with ➔ is how the output, i.e. the reading of the thermometer T, varies with time

- The **solution to the above differential equation** indicates how the output T changes with time t when there is a step input

$$\boldsymbol{T(t) = T_1 + (T_0 - T_1)e^{-t/\tau}}$$

Where:

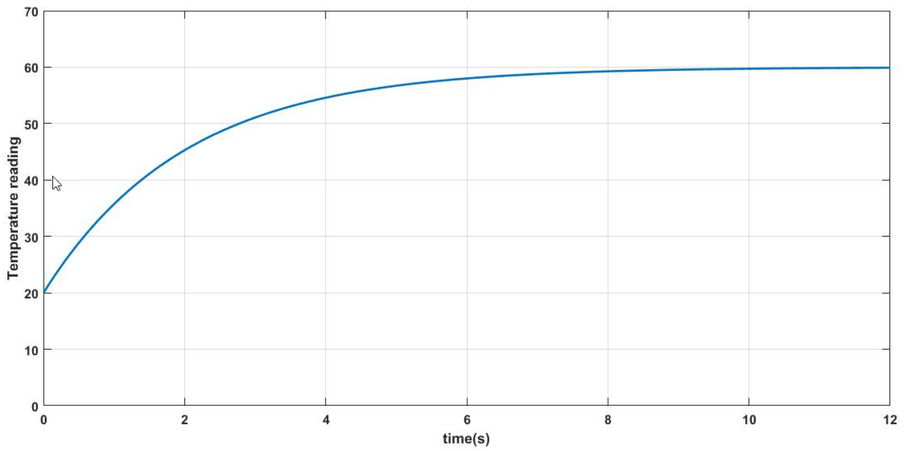
$\boldsymbol{\tau}$: time constant

Time constant is the time the system takes to reach 63.2% of its final value (see Table above).

$\boldsymbol{T_1}$: steady-state value – after sufficient time has elapsed for all transients to die away

Plot the response of the thermometer "First-order system" at temperature T_0 inserted into a liquid at a temperature T_1

```
T0 = 20;
T1 = 60;
t = 0:0.01:20;
tau = 2;
T2 = T1+(T0-T1)*exp(-t/tau);
plot(t,T2)
```

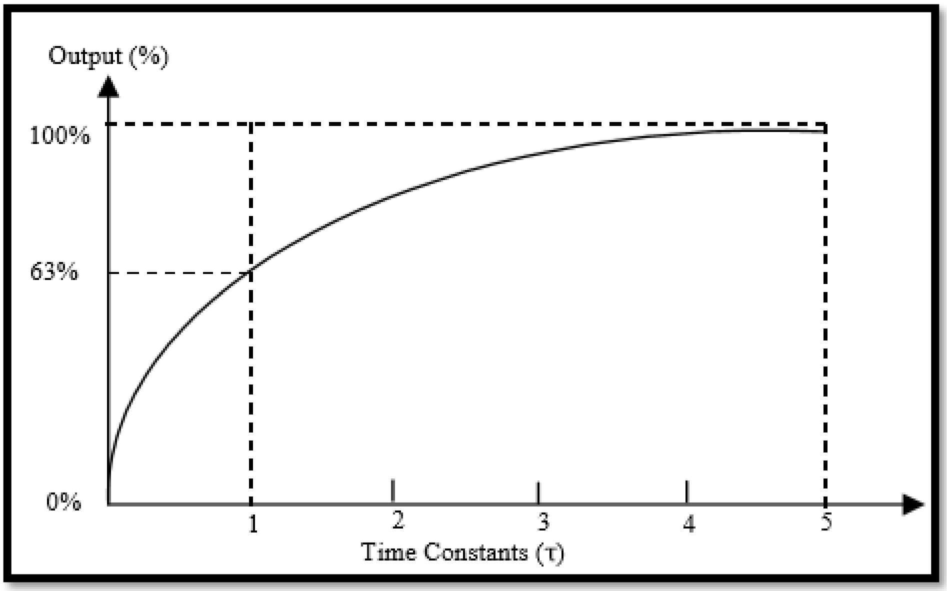


- The system response is called a "**first-order lag**" because the output **lags** behind the input, and the **differential equation** for the system shown in Figure below is a **linear first-order differential equation**.
- The **first-order lag** is the most common type of time element encountered in process control

✍ **Note:**

- If the thermometer is initially at zero temperature, i.e. $T_0 = 0$ °C

$$\boldsymbol{T(t) = T_1(1 - e^{-t/\tau})}$$



Response of first order system to step input

- After a time equal to one time constant the output reached about 63% of the way to the steady-state temperature, after a time equal to two time constant, the output reached about 86% of the way, after three-time constants about 95% and after about four time constants, it is virtual equal to the steady-state value
- The error at any time is the difference between what the thermometer is indicating and what the temperature actually is. Thus:

$$\text{Error} = T - T_1$$

And so:

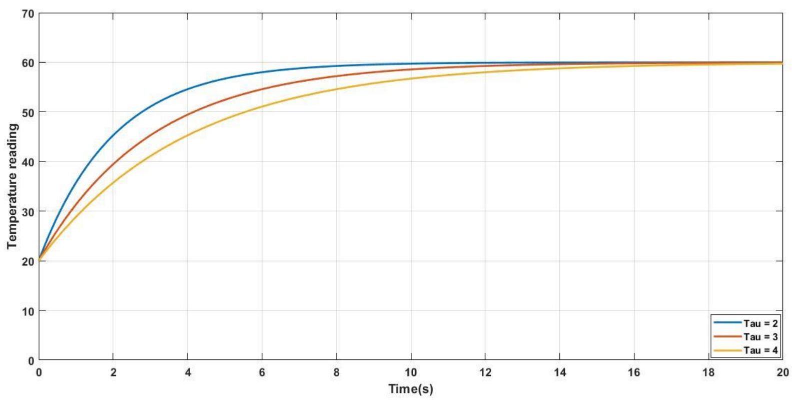
$$\text{Error} = (\boldsymbol{T_0} - \boldsymbol{T_1})\boldsymbol{e^{-t/\tau}}$$

- To illustrate the above, consider a thermometer which is indicating temperature of $20\,℃$ when it is suddenly immersed in a liquid at a temperature of $60\,℃$. If the thermometer has a time constant of $5\,s$ then the temperature T of the thermometer varies with time according to the equation

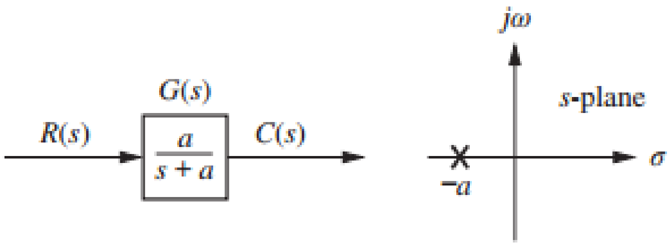
$$\boldsymbol{T(t) = T_1 + (T_0 - T_1)e^{-\frac{t}{\tau}} = 60 - 40\,e^{-t/\tau}}$$

After 5 s the thermometer reading will have reached about 63% of the way to the steady-state value, after 10 s about 86%, after 15 s, about 95% and after 20 s it is virtually at the steady state value. Thus after 5 s, the reading is 45.3 ℃, after 10 s it is 54.6 ℃, after 15 s it is 58.0 ℃

- The dynamic shape of their response to a step input is a function of their time constant. This system time constant, designated by the Greek letter **τ** (**tau**), is meaningful both in a physical and a mathematical sense. Physically, **it determines the shape of the response of a process or system to a step input**. Mathematically, it predicts, at any instant, the future time period that is required to obtain 63.2 percent of the change remaining. The response curve in Figure below illustrates the concept of a time constant by showing the response of a simple first-order system to a step input



A first-order system without zeros can be described by the transfer function shown. If the input is a unit step. Find the output of the system



$C(s) = R(s) * G(s)$

$$C(s) = \frac{a}{s+a} * R(s)$$

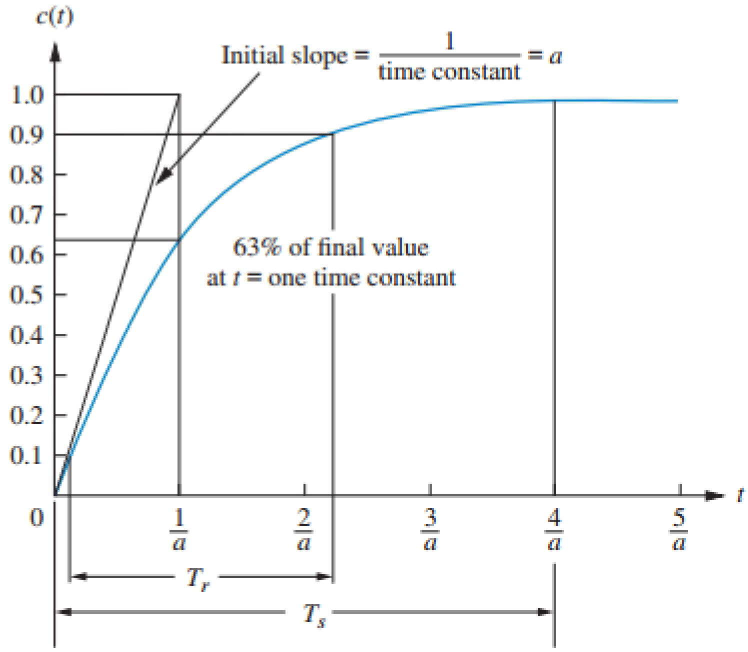
$$C(s) = \frac{a}{s+a} * \frac{1}{s} = \frac{1}{s} - \frac{1}{s+a}$$

$C(t) = 1 - e^{-at}$

Let us examine the significance of parameter a, the only parameter needed to describe the transient response. When $t = 1/a$,

$$C(t)|_{t=1/a} = 1 - e^{-1} = 1 - 0.37 = 0.63$$

We call $1/a$ the time constant of the response



The **reciprocal of the time constant** has the units (1/seconds), or frequency. Thus, we can call the parameter a the **exponential frequency**. Since the derivative of $\boldsymbol{e^{-at}}$ is $\boldsymbol{-a}$ when $t = 0$, a is the initial rate of change of the exponential at $t = 0$. **Thus, the time constant can be considered a transient response specification for a first-order system**, since it is related to the speed at which the system responds to a step input. The time constant can also be evaluated from the pole plot. Since the pole of the transfer function is at $\boldsymbol{a}$, we can say the pole is located at the *reciprocal* of the time constant, and the farther the pole from the imaginary axis, the faster the transient response.

- **Rise time:**

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

Rise time is found by solving $C(t) = 1 - e^{-at}$ for the difference in time at $c(t) = 0.9$ and $c(t) = 0.1$, Hence

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

- **Settling time:**

Settling time is defined as the time for the response to reach, and stay within, 2% of its final value. Letting $c(t) = 0.98$ in $C(t) = 1 - e^{-at}$ and solving for time, t, we find the settling time to be

$$T_s = \frac{4}{a}$$

- **First-order transfer functions via Testing**

Often it is not possible or practical to obtain a system's transfer function analytically. Perhaps the system is closed, and the component parts are not easily identifiable. Since the transfer function is a representation of the system from input to output, the system's step response can lead to a representation even though the inner construction is not known.

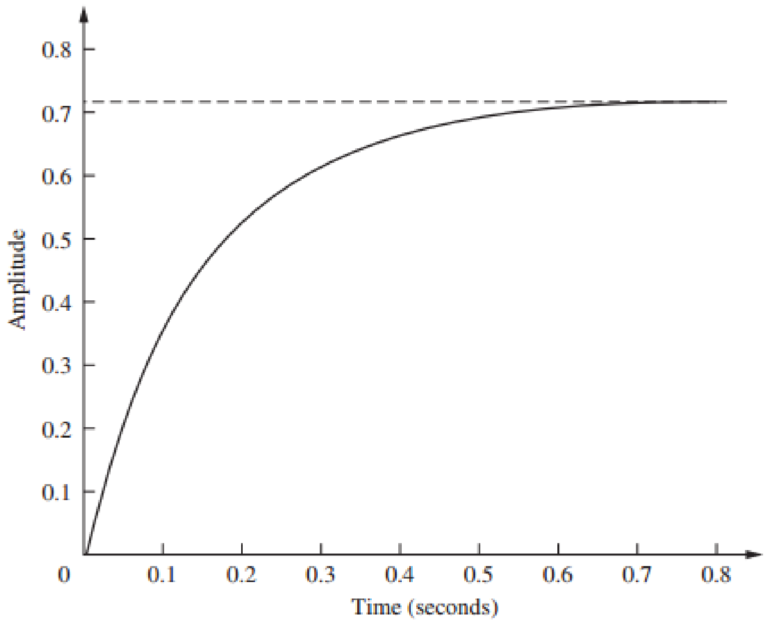
With a step input, we can **measure the time constant** and the **steady-state value**, from which the **transfer function can be calculated**.

Consider a simple first-order system, $G(s) = K/(s+a)$ whose step response is

$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{s+a}$$

If we can identify K and a from laboratory testing, we can obtain the transfer function of the system.

For example, assume the unit step response given in Figure below. We determine that it has the **first-order** characteristics we have seen thus far, such as **no overshoot** and **nonzero initial slope**.



From the response, we **measure** the **time constant**, that is, the time for the amplitude to reach 63% of its final value. Since the **final value is about $\mathbf{0.72}$**, the time constant is evaluated where the curve reaches $\mathbf{0{:}63 \times 0{:}72 = 0.45}$, or about $\mathbf{0.13}$ second. Hence, $\boldsymbol{a = \frac{1}{0.13} = 7.7}$

To find K, we realize from Eq.

$$\boldsymbol{C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{s+a}}$$

that the forced response reaches a steady-state value of $\boldsymbol{K/a = 0.72}$. Substituting the value of $\boldsymbol{a}$, we find $\boldsymbol{K = 5.54}$. Thus, the transfer function for the system is $\boldsymbol{G(s) = 5.54/(s+7.7)}$

> A system has a transfer function, $\boldsymbol{G(s) = \frac{50}{s+50}}$. Find the time constant, τ, settling time, T_s , and rise time, T_r.

$\tau = 0.02\ s$ $\qquad$ $T_s = 0.08\ s$ $\qquad$ $T_r = 0.044\ s$

- **Response of first order system to step input**

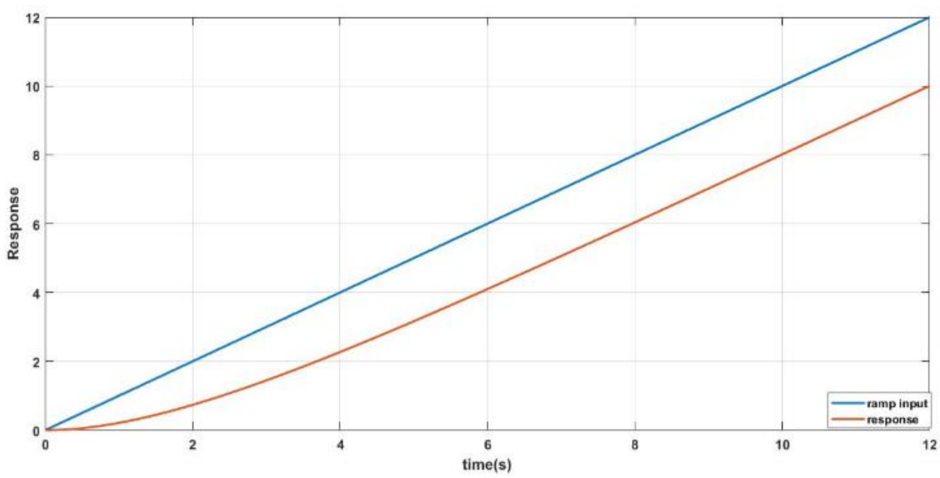
Since the Laplace transform of the unit-ramp function is $1/s^2$, we get the output of the 1st-order system as

$$C(s) = \frac{1}{s^2} * \frac{1}{\tau s + 1}$$

$$C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

$$c(t) = t - \tau + \tau e^{-t/\tau}$$

```
T0 = 0;
a = 1;    % slope
t = 0:0.01:12;
%%
T1 = T0+a*t;
tau = 2;
T2 = T1-a*tau+ a*tau*exp(-t/tau);
plot(t,T1,t,T2)
```



The error signal is $e(t)$ is then

$e(t) = r(t) - c(t)$

$e(t) = \tau(1 - e^{-t/\tau})$

As t approaches infinity, $e^{-t/\tau}$ approaches zero, and thus the error signal $e(t)$ approaches τ or

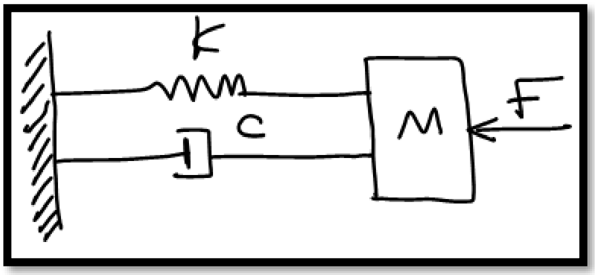
$$e(\infty) = \tau$$

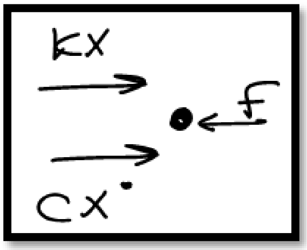
The smaller the time constant τ, the smaller the steady-state error in following the ramp input

■ Second-order system response

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 x$$

Compared to the simplicity of a first-order system, a **second-order system exhibits a wide range of responses that must be analyzed and described**. Whereas varying a first-order system's parameter simply changes the speed of the response, **changes in the parameters of a second-order system can change the form of the response**. For example, a second-order system can display characteristics much like a first-order system, or, depending on component values, **display damped** or **pure oscillations** for its transient response





$$\sum F = m\ddot{x}$$

$$F - c\dot{x} - kx = m\ddot{x}$$

$$(ms^2 + cs + k)X = F$$

$$\frac{X}{F} = \frac{1}{(ms^2 + cs + k)}$$

In the absence of damping ➔ $c = 0$

$$\frac{X}{F} = \frac{1}{ms^2 + k} = \frac{1/m}{s^2 + k/m}$$

$\frac{X}{F} = \frac{1/m}{s^2+k/m}$

there will be a term in t-domain that contains cos $\sqrt{\frac{k}{m}}t$

$$\therefore \quad \boldsymbol{\omega_n = \sqrt{\frac{k}{m}}}$$

mass m on the end of a spring freely oscillates with a natural frequency $\boldsymbol{\omega_n = \sqrt{\frac{k}{m}}}$

$$\because \quad \frac{X}{F} = \frac{1}{ms^2 + cs + k}$$

Divide by m

$$\therefore \quad \boldsymbol{\frac{X}{F} = \frac{1/m}{s^2 + \frac{c}{m}s + k/m}}$$

$$\because \quad \boldsymbol{\omega_n = \sqrt{\frac{k}{m}}} \qquad \therefore \quad \boldsymbol{{\omega_n}^2 = k/m}$$

Assume another term called $\zeta = \frac{c}{2\sqrt{mk}}$

Multiply $\boldsymbol{\omega_n}$ by ζ equation

$$\therefore \quad \zeta\boldsymbol{\omega_n} = \frac{c}{2m}$$

$$\therefore \quad 2\zeta\omega_n = \frac{c}{m}$$

$$\because \quad \frac{\boldsymbol{X}}{\boldsymbol{F}} = \frac{\mathbf{1}/\boldsymbol{m}}{\boldsymbol{s}^2 + \frac{\boldsymbol{c}}{\boldsymbol{m}}\boldsymbol{s} + \boldsymbol{k}/\boldsymbol{m}}$$

$$\boxed{\therefore \quad \frac{\boldsymbol{X}}{\boldsymbol{F}} = \frac{\mathbf{1}/\boldsymbol{m}}{\boldsymbol{s}^2 + \mathbf{2}\boldsymbol{\zeta\omega_n s} + \boldsymbol{\omega_n}^2}}$$

<u>If you are interested in step response of an instrument with second-order system</u>

Assume $f(t)$ is a step input with an amplitude F ➔ $f(t) = F$

$$\therefore \quad F(s) = \frac{F}{s}$$

$$\therefore \quad \boldsymbol{X} = \frac{\boldsymbol{F}/\boldsymbol{m}}{\boldsymbol{s}(\boldsymbol{s}^2 + \mathbf{2}\boldsymbol{\zeta\omega_n s} + \boldsymbol{\omega_n}^2)}$$

use partial fraction to get

$$\boldsymbol{X}(\boldsymbol{s}) = \frac{\boldsymbol{F}}{\boldsymbol{k}}\left[\frac{\mathbf{1}}{\boldsymbol{s}} - \frac{\boldsymbol{s} + \mathbf{2}\boldsymbol{\zeta\omega_n}}{\boldsymbol{s}^2 + \mathbf{2}\boldsymbol{\zeta\omega_n s} + \boldsymbol{\omega_n}^2}\right]$$

The time-domain solution to this equation depends on the value of $\boldsymbol{\zeta}$, so we have:

Underdamped case — $0 < \zeta < 1$

Report

Sketch the transient response of a 2nd order system to a unit step for different values of $\boldsymbol{\zeta}$ in the range of $0 < \boldsymbol{\zeta} < 1$ — hint: draw with respect to $\boldsymbol{\omega_n t}$

critically damped case — $\zeta = 1$

Report

Sketch the transient response of a 2nd order system to a unit step for $\boldsymbol{\zeta} = 1$ — hint: draw with respect to $\boldsymbol{\omega_n t}$

Overdamped case — $\zeta > 1$

Report

Sketch the transient response of a 2nd order system to a unit step for different values of $\boldsymbol{\zeta}$ in the range of $\boldsymbol{\zeta} > 1$ — hint: draw with respect to $\boldsymbol{\omega_n t}$

Characteristic equation:

The standard form of the second-order system

$$\boldsymbol{T.F = \frac{1/m}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}}$$

$\boldsymbol{\omega_n}$: undamped natural frequency (rad/s) and

$\boldsymbol{\zeta}$: Damping ratio

$\boldsymbol{\omega_d}$: damped natural frequency $\omega_n\sqrt{1-\zeta^2}$

$\boldsymbol{C/Cs}$ **equation**: $\boldsymbol{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$

This polynomial in s is called the **Characteristic Equation** and its ***roots will determine the system transient response.***

The dynamic behavior of the second-order system can then be described in terms of two parameters $\boldsymbol{\zeta}$ and $\boldsymbol{\omega_n}$.

- If $\boldsymbol{0 < \zeta < 1}$, the **closed-loop poles** are complex conjugates and lie in the **left-half s plane**. The system is then called **underdamped**, and the transient response is **oscillatory**.
- If $\boldsymbol{\zeta = 0}$, the transient response does not die out.
- If $\boldsymbol{\zeta = 1}$, the system is called **critically damped**.

- **Overdamped** systems correspond to $\boldsymbol{\zeta > 1}$.

Recall again that, the transfer function of the 2nd-order system is

$$\boldsymbol{T.F = \frac{1/m}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}}$$

And its roots can be obtained as

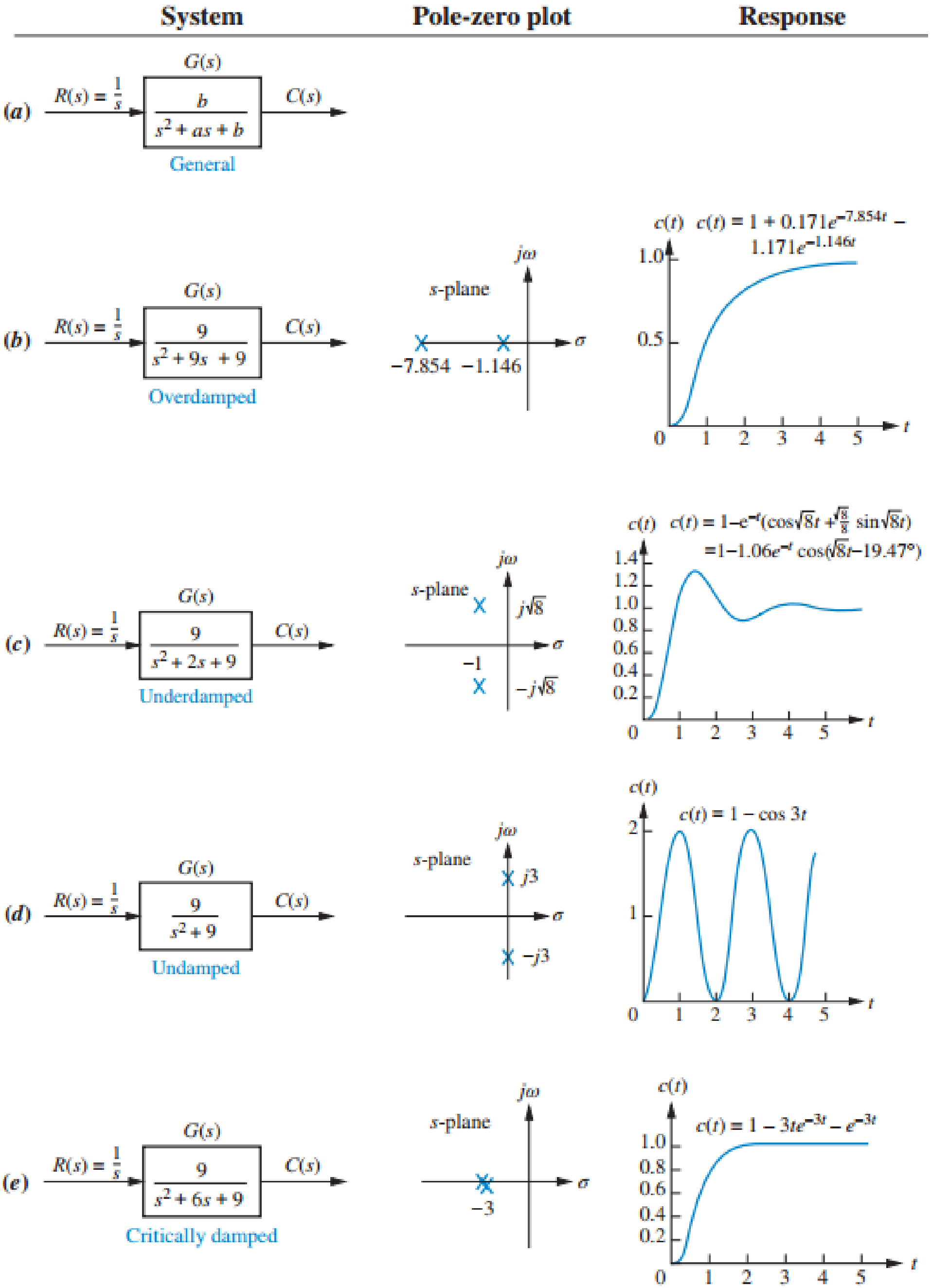
$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\boldsymbol{s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$

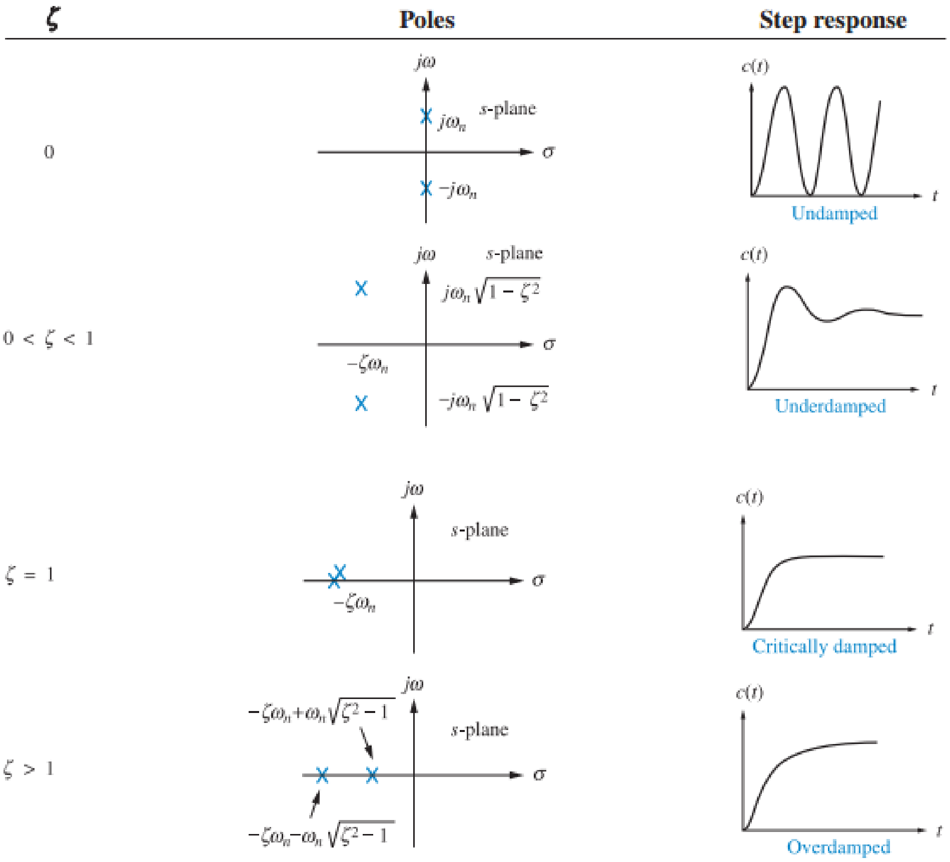
Depending on the values of the term $(b^2 - 4ac)$, called the ***discriminant***, may be positive, zero or negative which will make the roots real and unequal, real and equal or complex. This gives rise to the three different types of transient response described in Table below.

Transient behaviour of a second-order system

Discriminant	*Roots*	*Transient response type*
$b^2 > 4ac$	s_1 and s_2 real and unequal $(-\text{ve})$	Overdamped Transient Response
$b^2 = 4ac$	s_1 and s_2 real and equal $(-\text{ve})$	Critically Damped Transient Response
$b^2 < 4ac$	s_1 and s_2 complex conjugate of the form: $s_1, s_2 = -\sigma \pm \mathrm{j}\omega$	Underdamped Transient Response

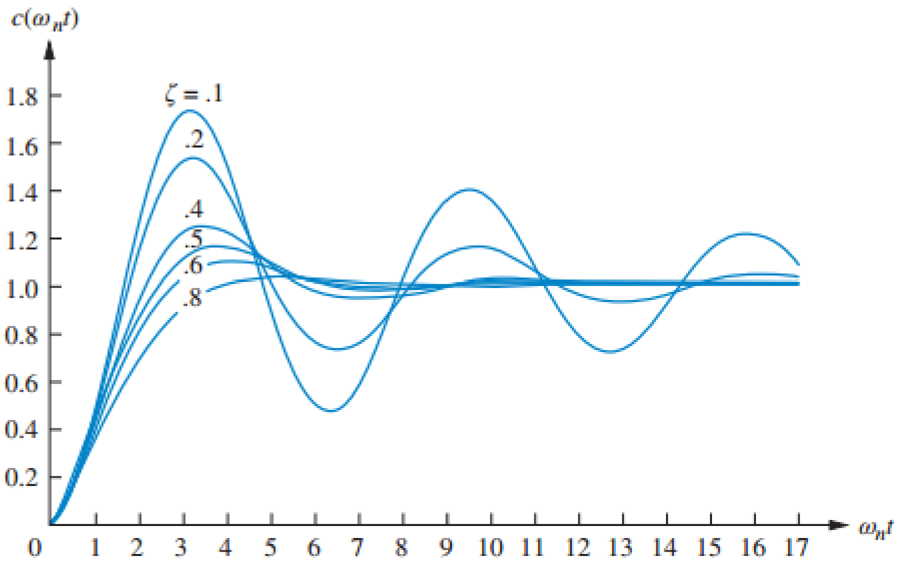


$C(s) = R(s)G(s)$, where $R(s) = 1/s$ followed by a partial-fraction expansion and the inverse Laplace transform.



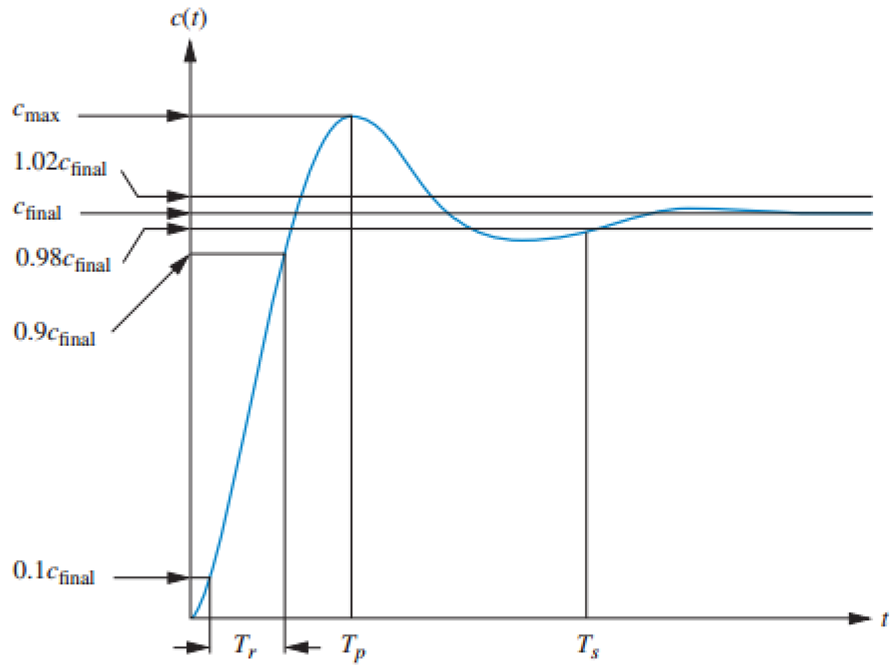
- Underdamped Second-Order Systems

let us analyze the step response of an *underdamped* second-order system. Not only will this response be found in terms of ζ and ω_n , but more specifications inherent to the underdamped case will be defined



We now see the relationship between the value of ζ and the type of response obtained: The lower the value of ζ, the more oscillatory the response. The natural frequency ω_n is a time-axis scale factor and does not affect the nature of the response other than to scale it in time.

We have defined two parameters associated with second-order systems, ζ and ω_n. Other parameters associated with the underdamped response are **rise time**, **peak time**, **percent overshoot**, and **settling time**. These specifications are defined as follows



Second-order underdamped response specifications

Rise time, T_r: The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

Peak time, T_p : The time required to reach the first, or maximum, peak.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Percent overshoot, $\%OS$: The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

$$\%OS = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Settling time, T_s: The time required for the transient's damped oscillations to reach and stay within 2% of the steady-state value

$$T_s = \frac{4}{\zeta \omega_n}$$

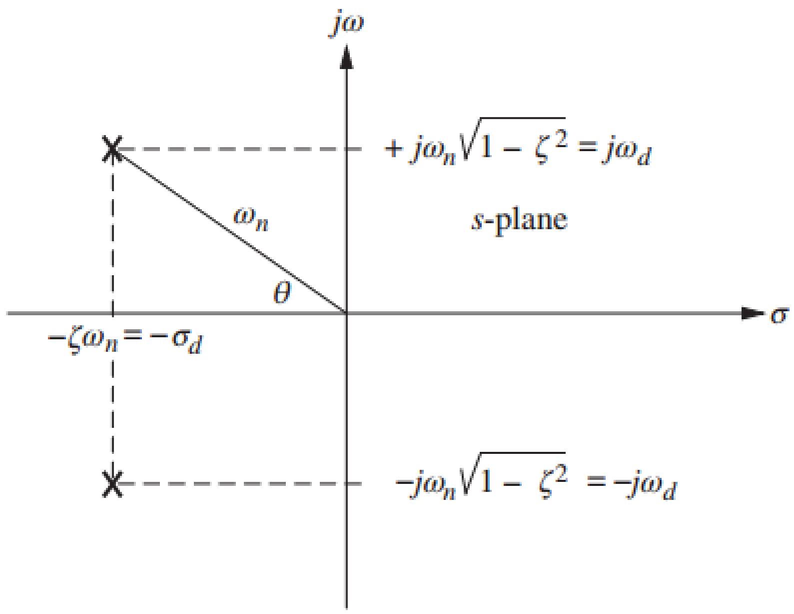
Notice that the definitions for settling time and rise time are basically the same as the definitions for the first-order response.

All definitions are also valid for systems of order higher than 2, although analytical expressions for these parameters cannot be found unless the response of the higher-order system can be approximated as a second-order system

Rise time, **peak time**, and **settling time** yield information about the speed of the transient response. This information can help a designer determine if the speed and the nature of the response do or do not degrade the performance of the system. For example, the speed of an entire computer system depends on the time it takes for a hard drive head to reach steady state and read data; passenger comfort depends in part on the suspension system of a car and the number of oscillations it goes through after hitting a bump.

Note:

We now have expressions that relate peak time, percent overshoot, and settling time to the natural frequency and the damping ratio. Now let us relate these quantities to the location of the poles that generate these characteristics.



Pole plot for an underdamped second-order system

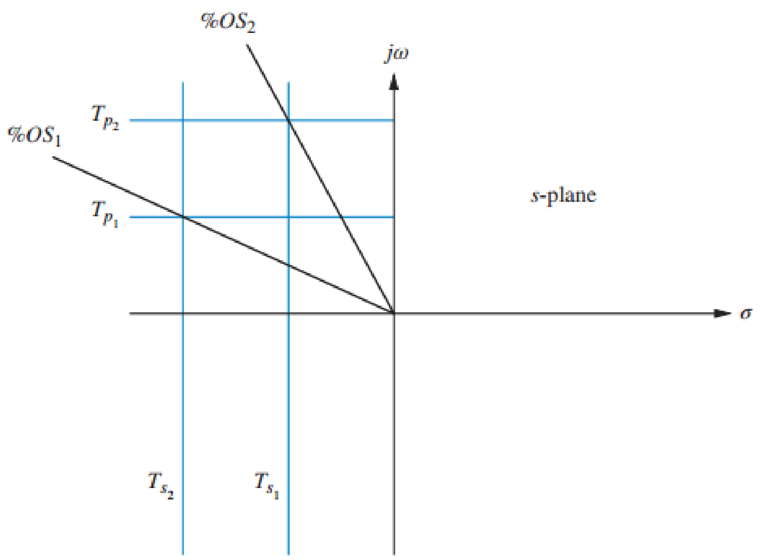
$$T_p = \frac{\pi}{\omega_d}$$

$$T_s = \frac{\pi}{\sigma_d}$$

$$\zeta = \cos\theta$$

Design criteria:

- Equation $T_p = \frac{\pi}{\omega_d}$ shows that T_p is inversely proportional to the imaginary part of the pole. Since horizontal lines on the s-plane are lines of constant imaginary value, they are also lines of constant peak time.
- Similarly, Eq. $T_s = \frac{\pi}{\sigma_d}$ tells us that settling time is inversely proportional to the real part of the pole. Since vertical lines on the s-plane are lines of constant real value, they are also lines of constant settling time.
- Finally, since $\zeta = \cos\theta$, radial lines are lines of constant ζ. Since percent overshoot is only a function of ζ, radial lines are thus lines of constant percent overshoot, $\%OS$. These concepts are depicted in Figure below, where lines of constant T_p, T_s, and $\%OS$ are labeled on the s-plane.



Poles, Zeros, and System Response

- The output response of a system is the sum of two responses: the forced response and the natural response.
- The **forced response** is also called the **steady-state response** or **particular solution**. The **natural response** is also called the **homogeneous solution**
- Although many techniques, such as **solving a differential equation or taking the inverse Laplace transform**, enable us to evaluate this output response, these techniques are laborious and time-consuming. Productivity is aided by analysis and design techniques that **yield results in a minimum of time**. If the technique is so rapid that we feel we derive the **desired result by inspection**, we sometimes use the attribute qualitative to describe the method. The **use of poles and zeros and their relationship to the time response of a system is such a technique**. Learning this relationship gives us a qualitative “handle” on problems. The concept of **poles and zeros**, fundamental to the analysis and design of control

- **Poles and zeros**

The closed loop transfer function $G(s)$ of a system can in general be represented by

$$G(s) = \frac{K(s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + \cdots + a_1 s + a_0)}{s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}$$

If the roots of the denominator and numerator are established then

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{s^n(s+p_1)(s+p_2)\dots(s+p_n)}$$

The values of s for which the numerator of the system transfer function is equal to zero are called **zeros** of the system

When the denominator of the system transfer function is set equal to zero, then values of $\boldsymbol{s}$ obtained, then values of $\boldsymbol{s}$ obtained are called the **poles**

Therefore, the roots of the numerator $\boldsymbol{-z_1, -z_2, \dots, -z_m}$ are called **zeros**

✍ **Note:** zeros are also any root of the numerator of the transfer function that are common to roots of the denominator.

if a factor of the numerator can be canceled by the same factor in the denominator, the root of this factor no longer causes the transfer function to become zero. In control systems, we **often refer to the root of the canceled factor in the numerator as a zero even though the transfer function will not be zero** at this value.

And the roots of the denominators $-\boldsymbol{p_1}, -\boldsymbol{p_2}, \ldots, -\boldsymbol{p_n}$ are called **poles**

Poles are the values of the $\boldsymbol{s}$ for which the transfer function is infinite

If the denominator is $(\boldsymbol{s}+\mathbf{5})$, then the transfer function is infinite when $\boldsymbol{s} = -\mathbf{5}$. Hence, $\boldsymbol{s} = -\mathbf{5}$ is the pole

✍ **Note:** any roots of the denominator of the transfer function that are common to roots of the numerator. if a factor of the denominator can be canceled by the same factor in the numerator, the root of this factor no longer causes the transfer function to become infinite.

In control systems, we **often refer to the root of the canceled factor in the denominator as a pole** even though the transfer function will not be infinite at this value.

Zeros and **poles** can be **real** or **complex quantities**

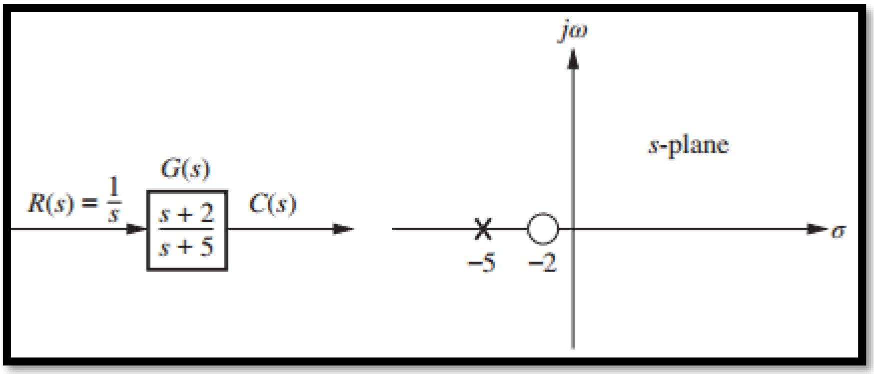
- **S-plane plots**

The Laplace variable s is a complex number; the values of s can be used to construct a plot. The **s-plane** is used to graphically represent the location of the poles and zeros of the transfer function

Given the transfer function

$$G(s) = \frac{(s+2)}{(s+5)}$$

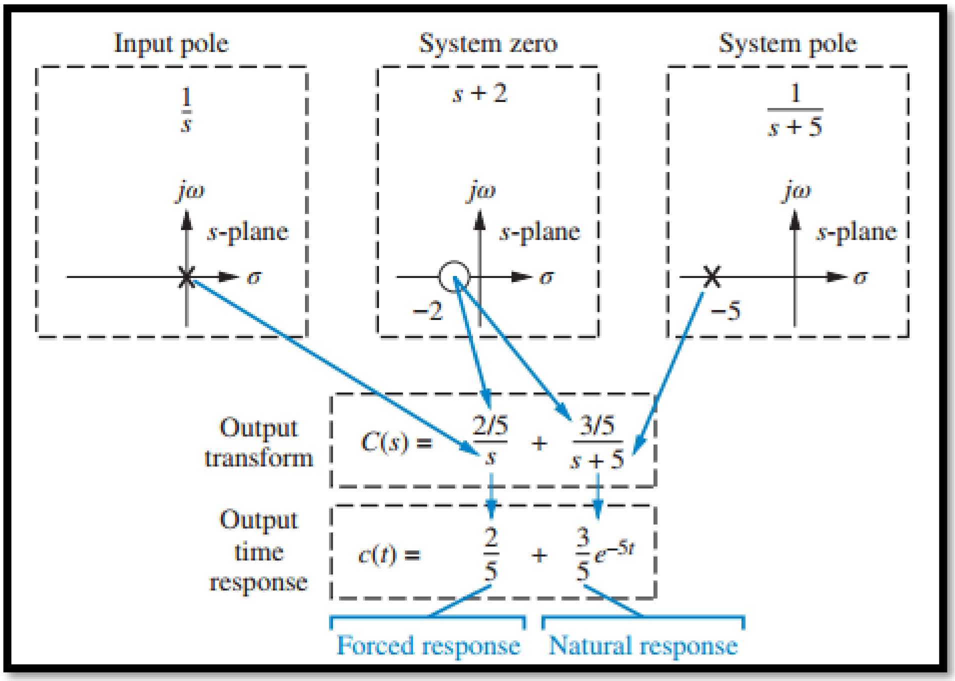
Pole exist at $s = -5$ and zero exists at $s = -2$



To show the properties of the poles and zeros, let us find the unit step response of the system. Multiplying the transfer function by a step function yields

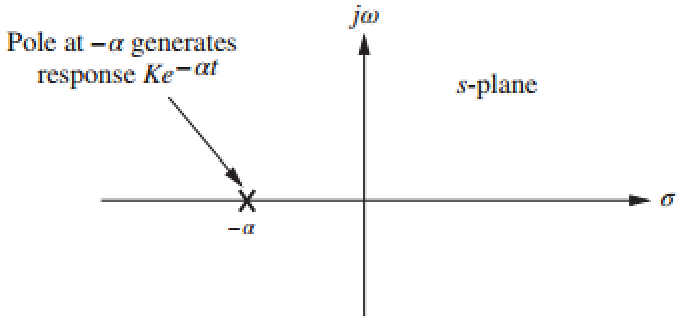
$$c(s) = \frac{(s+2)}{s(s+5)} = \frac{0.4}{s} + \frac{0.6}{s+5}$$

$$c(t) = 0.4 + 0.6e^{-5t}$$



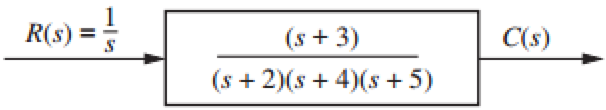
From the development summarized in Figure above, we draw the following conclusions:

- A **pole of the input function** generates the form of the **forced response** (that is, the pole at the origin generated a step function at the output).
- A pole of the transfer function generates the form of the natural response (that is, the pole at -5 generated e^{-5t}).
- A **pole on the real axis** generates an **exponential response** of the form $\boldsymbol{e^{-\alpha t}}$, where $\boldsymbol{\alpha}$ is the pole location on the real axis. **Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero** (again, the pole at 5 generated $\boldsymbol{e^{-5t}}$).



- The zeros and poles generate the amplitudes for both the forced and natural responses

Given the system below,

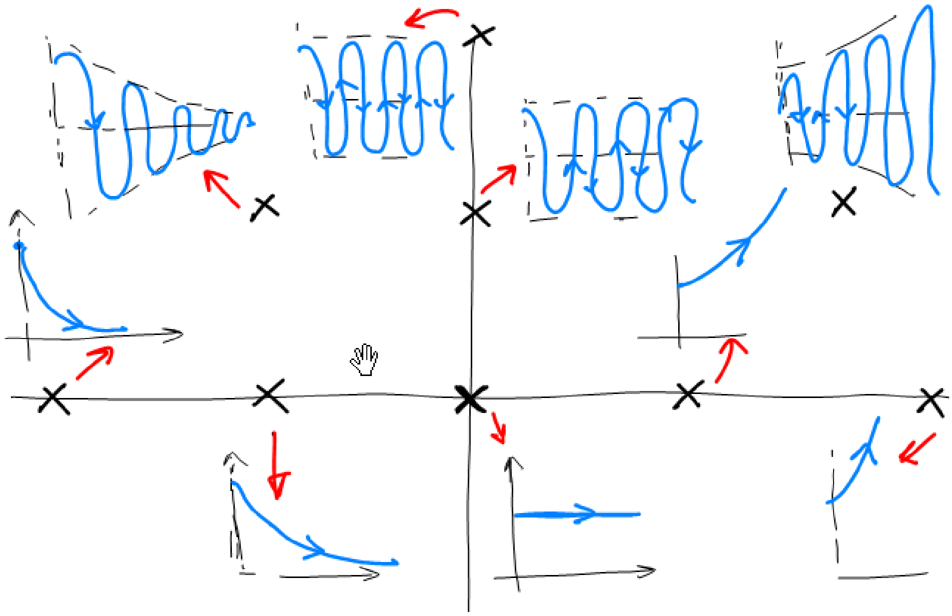


write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

$$c(s) = \frac{(s+3)}{s(s+2)(s+4)(s+5)} = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{s+4} + \frac{k_4}{s+5}$$

$$c(t) = k_1 + k_2 e^{-2t} + k_3 e^{-4t} + k_4 e^{-5t}$$

- Poles indicates response



Problem

1. For each of the following transfer functions, write, by inspection, the general form of the step response:
   a. $G(s) = \frac{400}{s^2+12s+400}$
   b. $G(s) = \frac{900}{s^2+90s+900}$
   c. $G(s) = \frac{225}{s^2+30s+225}$
   d. $G(s) = \frac{625}{s^2+625}$
2. Given the transfer function

   $$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

   Find $\boldsymbol{\zeta}$ and $\boldsymbol{\omega_n}$
3. Given the transfer function

   $$G(s) = \frac{100}{s^2 + 15s + 100}$$

   Find T_p, $\%OS$, T_s
4. Consider the system with $\zeta = 0.8$, $\omega_n = 12$ rad/sec

   Calculate

   Delay time, rise time, overshoot, and settling time
5. $G(s) = \frac{100}{s^2+s+100}$

   Find ω_n, ζ, ω_d
6. The transfer function of a system is given as $361/(s^2 + 16s + 361)$. Find the following for a unit step input:

(a) Undamped natural frequency

(b) Damping ratio

(c) Damped natural frequency

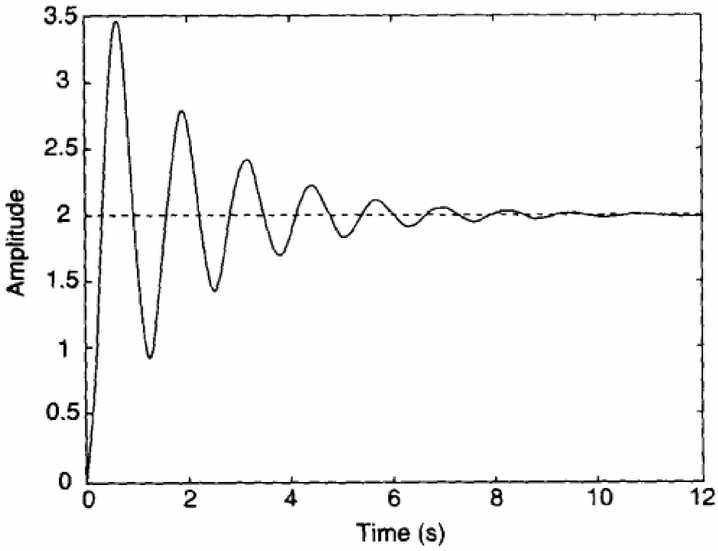
(d) Settling time

(e) Peak time

(f) Rise time

(g) Percentage overshoot

7. Figure below shows the unit-step response of a second-order system.



Determine the following from the plot:

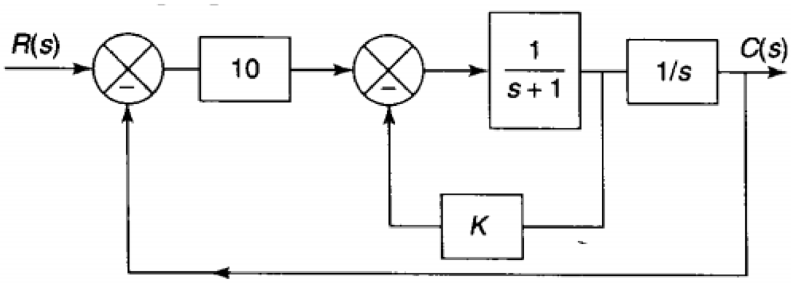
(a) The gain;

(b) Damping ratio,

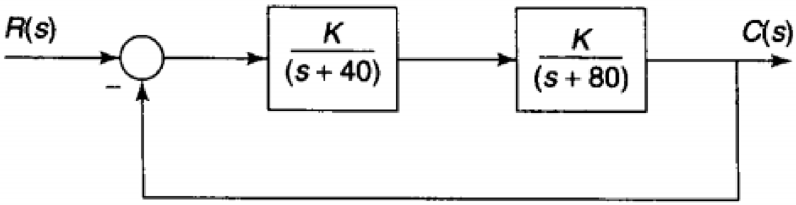
(c) Natural frequency,

(d) The transfer function

8. For the system shown below, determine the value of $\boldsymbol{K}$ such that the output signal will produce 10% overshoot in response to a step input



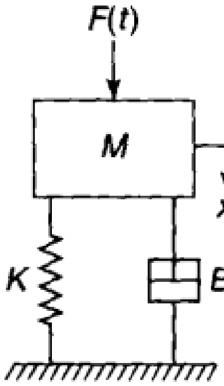
9. A servo mechanism is shown below

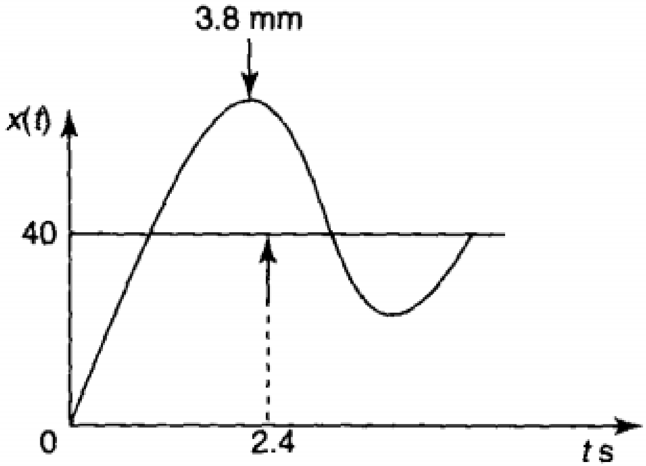


   (a) Calculate the value of $\boldsymbol{K}$ such that the natural frequency is 100 rad/sec

   (b) Determine the rise time

10. Figure below shows a mechanical vibratory system when $1\ kN$ force is applied to the system. The mass oscillates as shown. Determine the mass M, damping coefficient B and spring stiffness K of the system for this response curve


F(t)
M
x
K
B


3.8 mm
x(t)
40
0
2.4
t s

Chapter 07

Steady state error

Table of content

- Introduction
- MATLAB session
- SIMULINK

Introduction

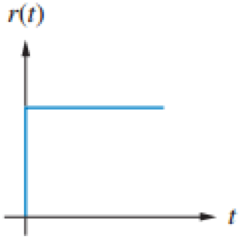
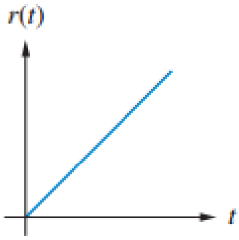
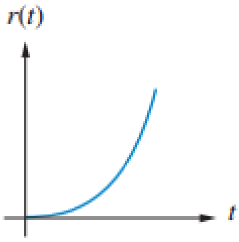
The static response of a system is the steady-state error that a system may exhibit between the controlled variable and the input

Whenever a system is distributed, or whenever an input is applied to a system, the system will be in **disturbed** state (**transient** state) for some time, and later on, it gets stabilized or attains a steady-state condition

The difference between the steady-state value and the reference value is known as the **steady state error** — it is a measure of the accuracy of a control system

> The **steady-state error** of a system response is defined as the discrepancy between the output and the reference input when the steady state ($t \to \infty$) is reached.

It should be pointed out that the **steady-state error may be defined for any test signal** such as a step-function, ramp-function, parabolic-function, or even a sinusoidal input

Waveform	Name	Physical interpretation
	Step	Constant position
	Ramp	Constant velocity
	Parabola	Constant acceleration

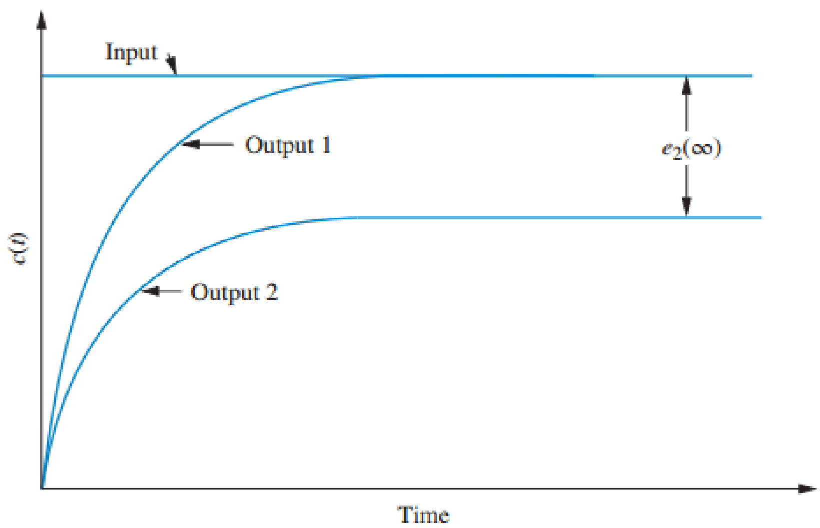
One of the objectives of most control systems is that the system output response follows a specific reference signal accurately in the steady state.

In the real world, **because of friction and other imperfections and the natural composition of the system**, the steady state of the output response seldom agrees exactly with the reference. Therefore, steady-state errors in control systems are almost unavoidable. **In a design problem**, one of the objectives is to keep the steady-state error to a minimum, or below a certain tolerable value, and at the same time the transient response must satisfy a certain set of specifications

Since we are concerned with the difference between the input and the output of a feedback control system after the steady state has been reached, **our discussion is limited to stable systems**, where the natural response approaches zero as Unstable systems represent loss of control in the steady state and are not acceptable for use at all.

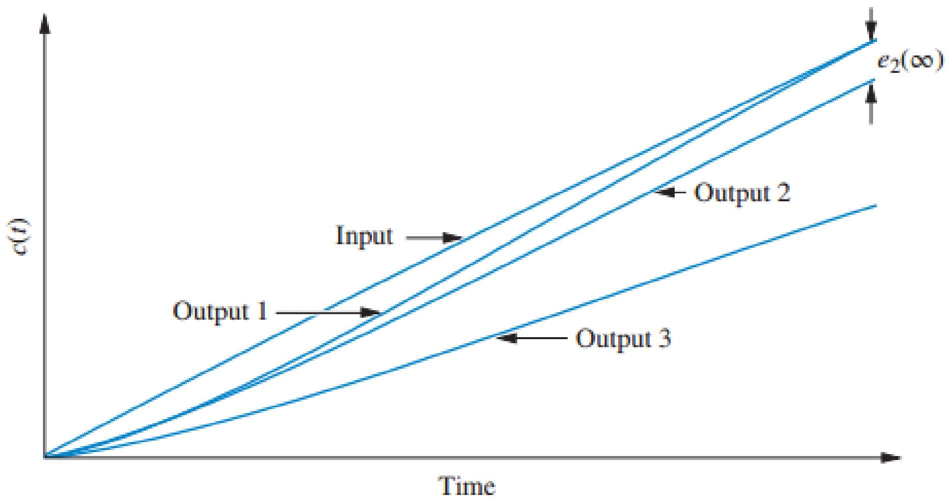
Thus, the engineer **must check the system for stability** while performing steady-state error analysis and design.

In Figure below a step input and two possible outputs are shown. Output 1 has zero steady-state error, and Output 2 has a finite steady-state error



Another example is shown below, where a ramp input is compared with Output 1, which has zero steady-state error, and Output 2, which has a finite steady-state error. Errors are measured vertically between the Input and Output 2 after the transients have died down.

For the ramp input another possibility exists. If the output's slope is different from that of the input, then Output 3, shown in Figure below, results. Here the steady-state error is infinite as measured vertically between the Input and Output 3 after the transients have died down, and t approaches infinity.



Many steady-state errors in control systems **arise from nonlinear sources**, such as backlash in gears or a motor that will not move unless the input voltage exceeds a threshold.

The steady- state errors we study here are errors that arise from the **configuration of the system itself** and the **type of applied input**.

Steady-state error (SSE) for open loop control system

$E(s) = R(s) - C(s)$

Also,

$\frac{C(s)}{R(s)} = G(s)$

$C(s) = R(s)G(s)$

$\therefore \quad E(s) = R(s) - R(s)G(s)$

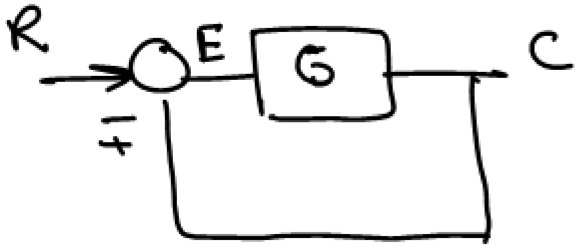
$E(s) = R(s)[1 - G(s)]$

$e_{ss} = \lim_{s \to 0} s\, E(s)$

$$\boldsymbol{e_{ss} = \lim_{s\to 0} s\, R(s)[1 - G(s)]}$$

Steady-state error (SSE) for closed loop control systems

Steady-state error can be calculated from a system's closed-loop transfer function, $T(s)$, or the open-loop transfer function, $G(s)$, for **unity feedback systems**



Here, $H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$C(s) = E(s)G(s)$$

$$\frac{E(s)G(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\boldsymbol{E(s) = \frac{R(s)}{1 + G(s)}}$$

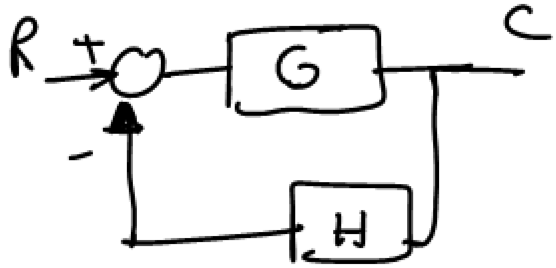
- Depends on input type, i.e. $\boldsymbol{R(s)}$
- Depends of system transfer function

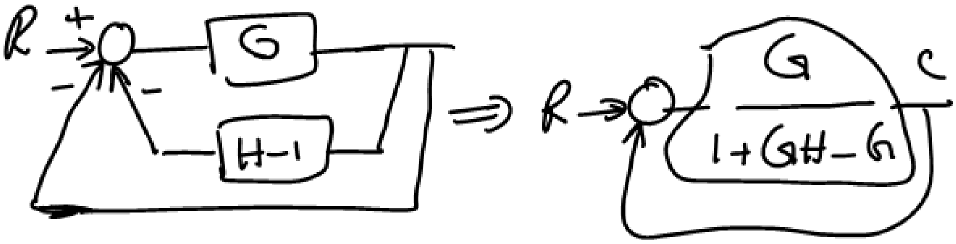
$\because \quad \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

$$\boldsymbol{e_{ss} = \lim_{s\to 0} s \frac{R(s)}{1+G(s)}}$$

✍ Note 01:

Any system can be described as unity feedback system to be able to obtain open-loop transfer function





Open loop transfer function

$$\boldsymbol{G_0(s) = \frac{G}{1+GH-G}}$$

✍ Note 02:

Transfer function

$$T = \frac{G_0}{1 + G_0}$$

$$T + TG_0 = G_0$$

$$T = G_0(1 - T)$$

$$G_0 = \frac{T}{1 - T}$$

So, given the transfer function for a system, you can get open loop transfer function G_0, then you can continue with other steps to get steady state error $\boldsymbol{e_{ss}}$

> Find the steady-state error for the transfer function
>
> $$T(s) = \frac{5}{(s^2 + 7s + 10)}$$
>
> and the input is a unit step

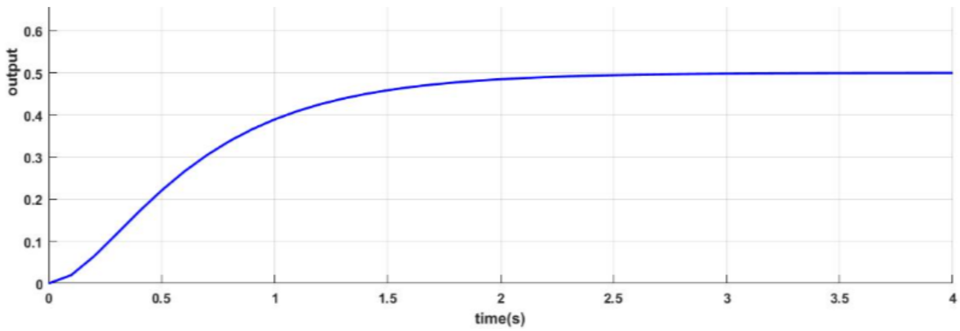
$$G_0 = \frac{5}{s^2 + 7s + 5}$$

$$e_{ss} = \lim_{s \to 0} s \frac{R(s)}{1 + G(s)}$$

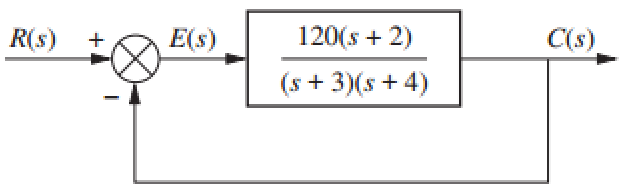
$$e_{ss} = \lim_{s\to 0} s \frac{1/s}{1+\frac{5}{s^2+7s+5}} = \lim_{s\to 0} \frac{s^2+7s+5}{s^2+7s+10}$$

$$e_{ss} = 0.5$$

```
a = [5];
b = [1 7 10];
sys = tf(a,b);
t = 0:0.1:4;
y = step(sys,t);
plot(t,y);
```



Find the steady state error for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown



$$G_0 = \frac{120(s+2)}{(s+3)(s+4)}$$

a. e_{ss} for input $5u(t)$

$$e_{ss} = \lim_{s\to 0} s \frac{5/s}{1 + \frac{120(s+2)}{(s+3)(s+4)}}$$

$$= \lim_{s\to 0} 5 * \frac{(s+3)(s+4)}{(s+3)(s+4) + 120(s+2)}$$

$$= 5 * \frac{12}{12 + 240} = 0.238$$

Also, for unit-step input, e_{ss} can be calculated from

$$e_{ss} = \lim_{s\to 0} s \frac{R(s)}{1 + G(s)} = \lim_{s\to 0} s \frac{1/s}{1 + G(s)}$$

$$e_{ss} = \frac{1}{1 + \lim_{s\to 0} G(s)}$$

For the numerical example above:

$$e_{ss} = \frac{5}{1 + \lim_{s\to 0} G(s)}$$

$$e_{ss} = \frac{5}{1 + \lim_{s\to 0} \frac{120(s+2)}{(s+3)(s+4)}} = \frac{5}{1 + \frac{120(2)}{(3)(4)}} = \frac{5}{21} = 0.238$$

b. e_{ss} for input $5tu(t)$

for unit-ramp input, e_{ss} can be calculated from

$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)} = \lim_{s\to 0} s\frac{1/s^2}{1+G(s)}$$

$$e_{ss} = \frac{1}{\lim_{s\to 0} sG(s)}$$

For the numerical example above:

$$e_{ss} = \frac{5}{\lim_{s\to 0} s\frac{120(s+2)}{(s+3)(s+4)}} = \infty$$

c. e_{ss} for input $5tu(t)$

for ramp input, e_{ss} can be calculated from

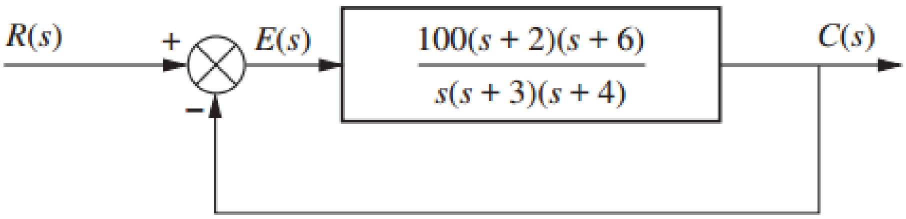
$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)} = \lim_{s\to 0} s\frac{2/s^3}{1+G(s)}$$

$$e_{ss} = \frac{2}{\lim_{s\to 0} s^2G(s)}$$

For the numerical example above:

$$e_{ss} = \frac{10}{\lim_{s\to 0} s^2\frac{120(s+2)}{(s+3)(s+4)}} = \infty$$

Find the steady state error for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown



(a) Steady-state error for $5\,u(t)$

$$e_{ss} = \frac{sR(s)}{1 + \lim\limits_{s\to 0} G(s)}$$

$$e_{ss} = \frac{5}{1 + \lim\limits_{s\to 0} G(s)}$$

$$e_{ss} = \frac{5}{1 + \lim\limits_{s\to 0} \dfrac{100(s+2)(s+6)}{s(s+3)(s+4)}}$$

$$e_{ss} = 0$$

(b) Steady-state error for $5\,tu(t)$

$$e_{ss} = \frac{5}{\lim\limits_{s\to 0} sG(s)}$$

$$e_{ss} = \frac{5}{\lim\limits_{s\to 0} s\dfrac{100(s+2)(s+6)}{s(s+3)(s+4)}}$$

$$e_{ss} = \frac{5}{100 * \frac{2*6}{3*4}} = 0.05$$

(c) Steady-state error for $5\ t^2 u(t)$

$$e_{ss} = \frac{10}{\lim_{s\to 0} s^2 G(s)}$$

$$e_{ss} = \frac{10}{\lim_{s\to 0} s^2 \frac{100(s+2)(s+6)}{s(s+3)(s+4)}} = \infty$$

A unity feedback system has the following forward transfer function

$$\frac{10(s+20)(s+30)}{s(s+25)(s+35)}$$

Find the steady-state error for the following inputs: $15\ u(t)$,$15t\ u(t)$, $15t^2\ u(t)$

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input

Control systems may be classified according to their **ability to follow step inputs, ramp inputs, parabolic inputs**, and so on. This is a reasonable classification scheme, because actual inputs may frequently be considered combinations of such inputs.

The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

If $G_0(s)$ can be written as

$$G_0(s) = \frac{k(s+z_1)(s+z_1)\dots(s+z_m)}{s^n(s+p_1)(s+p_2)\dots\left(s+p_q\right)}$$

$\boldsymbol{n}$: system type

$$G(s)H(s) = \frac{5(s+1)}{s^2(s+2)}$$

System type is 2

Note that this classification is different from that of the order of a system. **As the type number is increased**, **accuracy is improved**; however, **increasing the type number aggravates the stability problem**. A **compromise** between **steady-state accuracy** and **relative stability** is always necessary

- Static Error Constants

The static position error constant K_P "**use it when step input applied**" is defined by

$$K_P = \lim_{s\to 0} G(s) = G(0)$$

Thus, the steady-state error in terms of the static position error constant K_P is given by

$$e_{ss} = \frac{1}{1 + K_P}$$

The static velocity error constant K_v "**use it when ramp input applied**" is defined by

$$K_v = \lim_{s\to 0} sG(s)$$

Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

The static acceleration error constant K_a "**use it when parabolic input applied**" is defined by

$$K_a = \lim_{s\to 0} s^2 G(s)$$

Thus, the steady-state error in terms of the static velocity error constant K_a is given by

$$e_{ss} = \frac{1}{K_a}$$

The **static error constants** defined above are figures of merit of control systems. The higher the constants, the smaller the steady-state error.

In a given system, the output may be the position, velocity, pressure, temperature, or the like. The physical form of the output, however, is immaterial to the present analysis. Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on. This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on

The error constants K_P, K_v, and K_a describe the ability of a unity-feedback system to reduce or eliminate steady-state error. Therefore, they are indicative of the steady-state performance. It is generally desirable to increase the error constants, while maintaining the transient response within an acceptable range. It is noted that to improve the steady state performance we can increase the type of the system by adding an integrator or integrators to the feedforward path. This, however, introduces an additional stability problem.

Problems:

1. When the closed-loop system involves a numerator dynamic, the unit-step response curve may exhibit a large overshoot. Obtain the unit-step response of the following system with MATLAB:

$$\frac{C(s)}{R(s)} = \frac{10s + 4}{s^2 + 4s + 4}$$

   Obtain also the unit-ramp response with MATLAB

2. Consider a unity-feedback control system with the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{Ks + b}{s^2 + as + b}$$

   Determine the open-loop transfer function $G(s)$. Show that the steady-state error in the unit-ramp response is given by

$$e_{ss} = \frac{1}{K_v} = \frac{a - K}{b}$$

3. Consider a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(Js + B)}$$

   Discuss the effects that varying the values of K and B has on the steady-state error in unit-ramp response. Sketch typical unit-ramp response curves for a small value, medium value, and large value of K, assuming that B is constant.

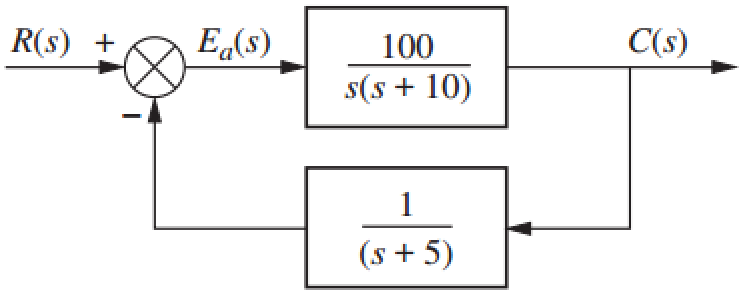
4. A unity feedback system has the following forward transfer function:

$$G(s) = \frac{K(s+12)}{(s+14)(s+18)}$$

Find the value of $\boldsymbol{K}$ to yield a 10% error in the steady state

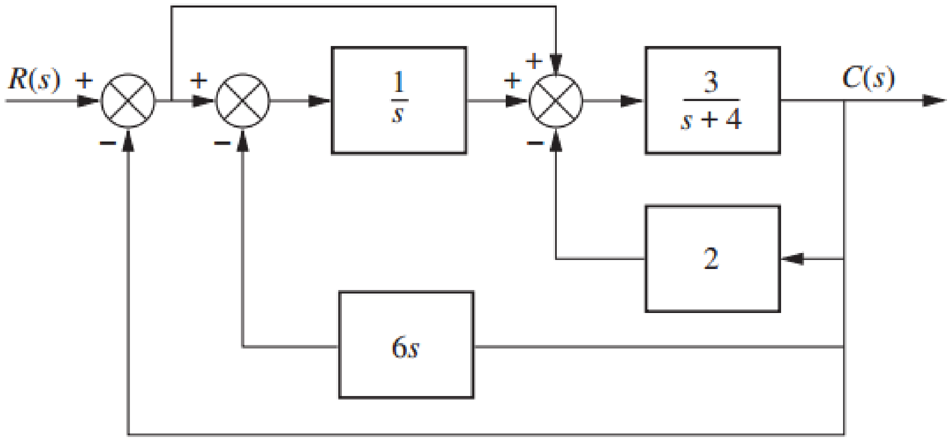
Ans. 189

5. For the system shown in Figure, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same

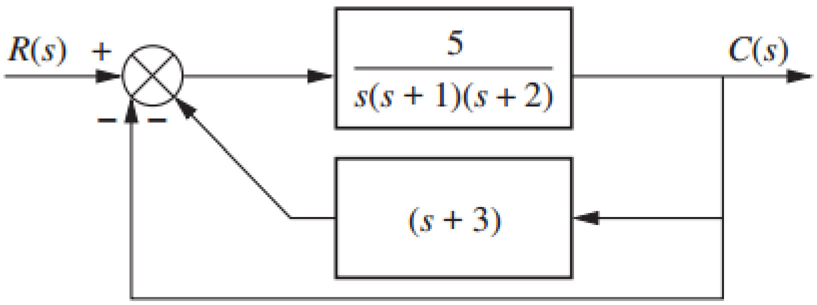


6. For the system shown in Figure, what steady-state error can be expected for the following test inputs:

$10u(t)$, $10tu(t)$, $10t^2u(t)$



7. For the system shown in Figure



(a) Find K_P, K_v, and K_a

(b) Find the steady-state error for an input of $50u(t)$, $50tu(t)$, $50t^2u(t)$

(c) State the system type.

Chapter 08

Stability analysis

<u>Table of content</u>

- Introduction
- MATLAB session
- SIMULINK

- **Introduction**

A **stable system** is defined as a system with a **bounded** (**limited**) system response. That is, if the system is subjected to a bounded input or disturbance and the **response is bounded in magnitude**, the system is said to be **stable**.

> A **stable system** is a dynamic system with a bounded response to a bounded input

A **necessary and sufficient condition** for a feedback system to be stable is that **all the poles of the system transfer function have negative real parts**

A system is not stable if not all the roots are in the left-hand plane

If the **characteristic equation** has **simple roots on the imaginary axis** ($j\omega$-axis) **with all other roots in the left half-plane**, the steady-state output will be **sustained oscillations**

For an **unstable system**, the characteristic equation has at least one root in the right half of the s-plane or repeated $j\omega$ roots; for this case, the output will become unbounded for any input

Consider that the characteristic equation of a linear time-variant **SISO** system is of the form

$$C/cs = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

where all the coefficients are real. **To ensure the last equation does not have roots with positive real parts, it is necessary (but not sufficient**) that the following conditions hold:

- All the coefficients of the equation have the same sign
- None of the coefficients vanish

However, these conditions are not sufficient, for it is quite possible that an equation with all its coefficients nonzero and of the same sign still **may not have all the roots in the left half of the s-plane**

- **ROUTH'S STABILITY CRITERION**

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system?

it was stated that a control system is stable if and only if all closed-loop poles lie in the left-half s plane. Most linear closed-loop systems have closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

where the a's and b's are constants and $\boldsymbol{m} \leq \boldsymbol{n}$. A simple criterion, known as **Routh's stability criterion**, enables us to determine the number of closed-loop poles that lie in the right-half

s plane without having to factor the denominator polynomial. (The polynomial may include parameters that MATLAB cannot handle)

consider

$$C/cs = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

A necessary and sufficient condition for stability if and only if all the elements in the first column of the Routh array have no sign change

s^n	1	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	b_1	b_2	b_3	...
s^{n-3}	c_1	c_2	c_3	...
...	...	...	...	...
s^2	*	*		
s	*			
s^0	*			

$$b_1 = \frac{a_1 * a_2 - a_3 * 1}{a_1}$$

$$b_2 = \frac{a_1 * a_4 - a_5 * 1}{a_1}$$

$$b_3 = \frac{a_1 * a_6 - a_7 * 1}{a_1}$$

$$c_1 = \frac{b_1 * a_3 - b_2 * a_1}{b_1}$$

$$c_2 = \frac{b_1 * a_5 - b_3 * a_1}{b_1}$$

The number of roots in RHS is equal to the number of sign change in the 1st column

Example:

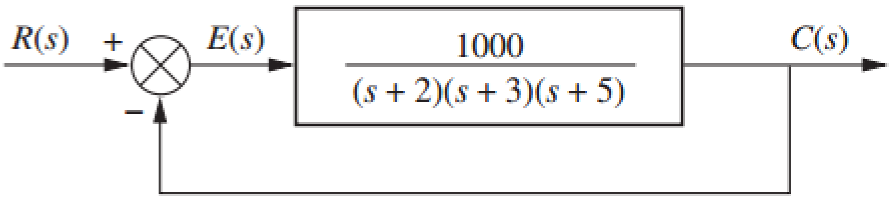
Given that the closed loop transfer function is

$$\frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\begin{array}{llll}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & \dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 & \dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 & \dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0 \\
s^1 & \dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0 & \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0 \\
s^0 & \dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 & \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0 & \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0
\end{array}$$

Example:

Make the Routh table for the system shown



$$TF = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

s^3	1	31	0
s^2	10	1030	
s^1			
s^0			

For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below.

s^3	1	31	0
s^2	1	103	0
s^1	-72	0	0
s^0	103		

The number of sign change is 2, so there are two roots on the RHS.

The system is unstable

```
a = [1 10 31 1030];
roots(a)
```

-13.4136 + 0.0000i

1.7068 + 8.5950i

1.7068 - 8.5950i

Example:

C/cs. = $s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$

s^6	1	3	1	4
s^5	4	2	4	
s^4	2.5	0	4	
s^3	2	-2.4	0	
s^2	3	4		
s^1	-5.066			
s^0	4			

1	+
4	+
2.5	+
2	+
3	+

-5.066	-
4	+

The sign change is two, so we have two roots on the RHS

-3.2644 + 0.0000i

0.6797 + 0.7488i

0.6797 - 0.7488i

-0.6046 + 0.9935i

-0.6046 - 0.9935i

-0.8858 + 0.0000i

Example:

Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane

C/cs. = $3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$

s^7	3	6	7	2
s^6	9	4	8	6
s^5				
s^4				
s^3				
s^2				
s^1				

s^0	4

Example:

C/cs. = $s^4 + 2s^3 + 3s^2 + 4s + 5$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6		
s^0	5		

1	+	Two sign changes 2 roots on RHS 2 on the LHS
2	+	
1	+	
-6	-	
5	+	

Example:

C/cs. = $a_2s^2 + a_1s + a_0$

s^2	a_2	a_0
s^1	a_1	0

s^0	a_0

Therefore, the requirement for a **stable second-order** system is simply that **all the coefficients be positive or all the coefficients be negative**

Example:

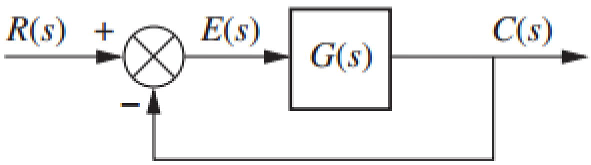
C/cs. = $a_3s^3 + a_2s^2 + a_1s + a_0$

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$\dfrac{a_2a_1 - a_0a_3}{a_2}$	
s^0	a_0	

For the third-order system to be stable, it is necessary and sufficient that the coefficients be positive and $a_2a_1 > a_0a_3$. The condition when $a_2a_1 = a_0a_3$ results in a marginal stability case, and one pair of roots lies on the imaginary axis in the s-plane

Problem:

For the unity feedback system of Figure



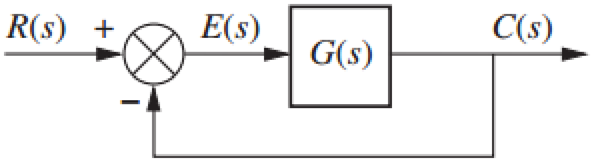
With

$$G(s) = \frac{K(s+6)}{s(s+1)(s+4)}$$

Determine the range of K to ensure stability

Problem:

For the unity feedback system of Figure



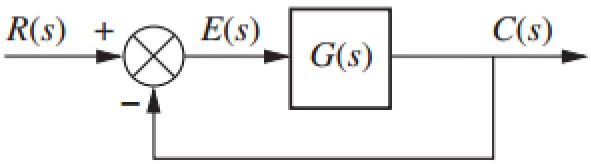
With

$$G(s) = \frac{K(s+3)(s+5)}{(s-2)(s-4)}$$

Determine the range of K to ensure stability

Problem:

Determine if unity feedback system of fig. below with



With

$$G(s) = \frac{K(s^2+1)}{(s+1)(s+2)}$$

Can be unstable

Special case:

If only the first element in one of the rows is zero, then we can replace the zero with a small positive constant $\epsilon > 0$ and proceed as follows:

(1) If the sign of the coefficient above the zero is the same as the one below it, it indicates that there are a pair of imaginary roots

(2) If the sign of the coefficient above the zero is opposite that below it, there is one change

Example:

C/cs. = $s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$

s^5	1	2	6
s^4	3	6	9
s^3	$\cancel{0}\ \ \epsilon$	3	
s^2	$6 - 9/\epsilon$	9	
s^1	$3 - \dfrac{9\epsilon^2}{6\epsilon - 9}$		

s^0	9

Take the limit as $\epsilon \rightarrow 0$

1	+		
3	+		
ϵ	ϵ	1 sign change	
$6-9/\epsilon$	-		Two sign changes
$3-\dfrac{9\epsilon^2}{6\epsilon-9}$	+	1 sign change	
9	+		

You may check by MATLAB

-2.9043 + 0.0000i

0.6567 + 1.2881i

0.6567 - 1.2881i

-0.7046 + 0.9929i

-0.7046 - 0.9929i

Example:

C/cs. = s^3+2s^2+s+2

s^3	1	1
s^2	2	2

s^1	$\cancel{0}\ \epsilon$
s^0	2

Pair of imaginary roots

-2.0000 + 0.0000i

0.0000 + 1.0000i

0.0000 - 1.0000i

Example:

C/cs. = $s^3 - 3s + 2$

s^3	1	-3	Two sign
s^2	0	2	changes
s^1	$-3-2/\epsilon$		2 roots on
s^0	2		RHS

✍ Note:

$$s^3 - 3s + 2 = (s-1)^2(s+2)$$

Example:

C/cs. = $s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$

s^5	1	11	28	Two complex poles on imaginary axis
s^4	5	23	12	
s^3	6.4	25.6		
s^2	**3**	12		
s^1	$\cancel{0}$ ϵ			
s^0	**12**			

Special case:

When an entire row of the Routh array is zero. This indicates that there are complex conjugate pairs of roots that are mirror images of each other with respect to the imaginary axis

Example:

C/cs. = $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50$

s^5	1	24	-25	
s^4	2	48	-50	Auxiliary equation
s^3	0	0	0	Row = 0
s^2	**3**	12		
s^1	$\cancel{0}$ ϵ			
s^0	**12**			

Auxiliary equation

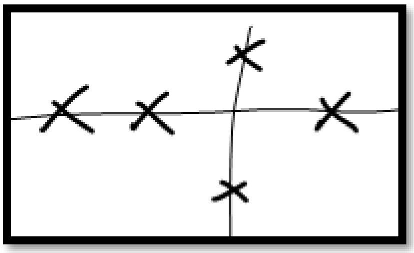
P(s) = $2s^4 + 48s^2 - 50$

$$\frac{dP}{ds} = 8s^3 + 96s$$

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.67		
s^0	-50		

s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.67		
s^0	-50		

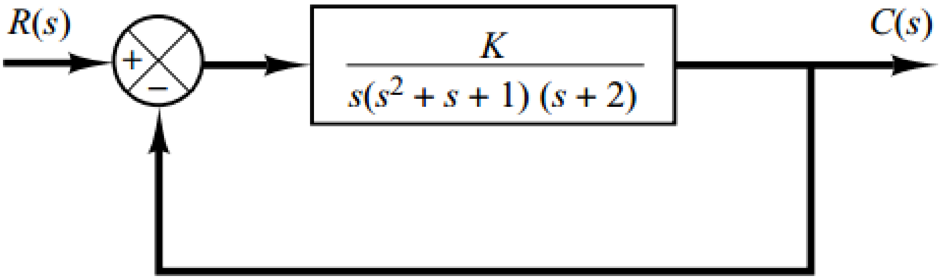
One sign change so we have one root on the RHS, so we have 1 on LHS→ 4 - 2 = 2 → imaginary



5 – 4 → 1 → LHS

Example:

Consider the system shown in Figure. Let us determine the range of K for stability. The closed-loop transfer function is



$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

$$\begin{array}{l|lll|}
s^4 & 1 & 3 & K \\
s^3 & 3 & 2 & 0 \\
s^2 & 7/3 & K & \\
s^1 & 2 - \frac{9}{7}K & & \\
s^0 & K & &
\end{array}$$

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

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